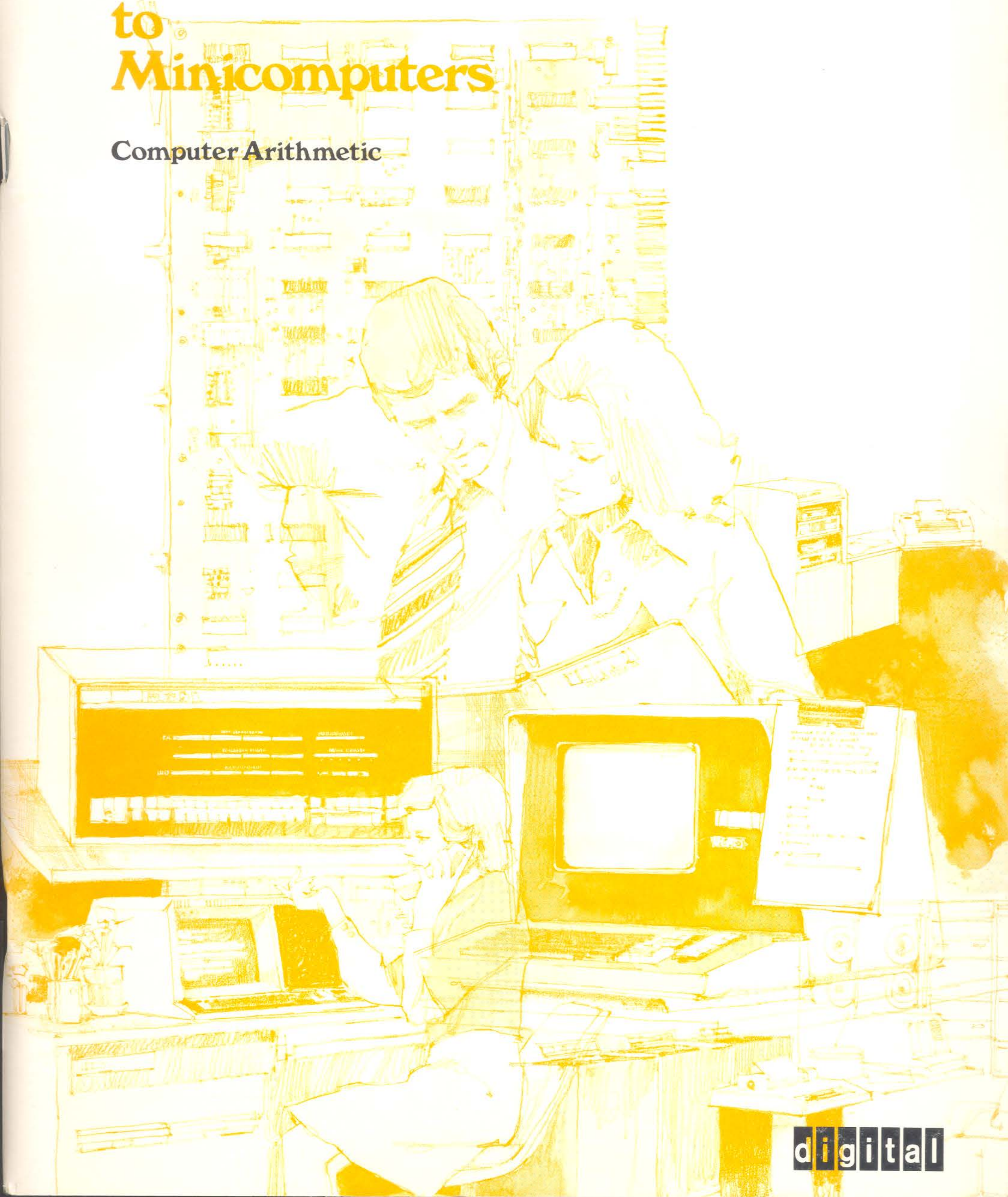
Introduction to Minicomputers

Computer Arithmetic



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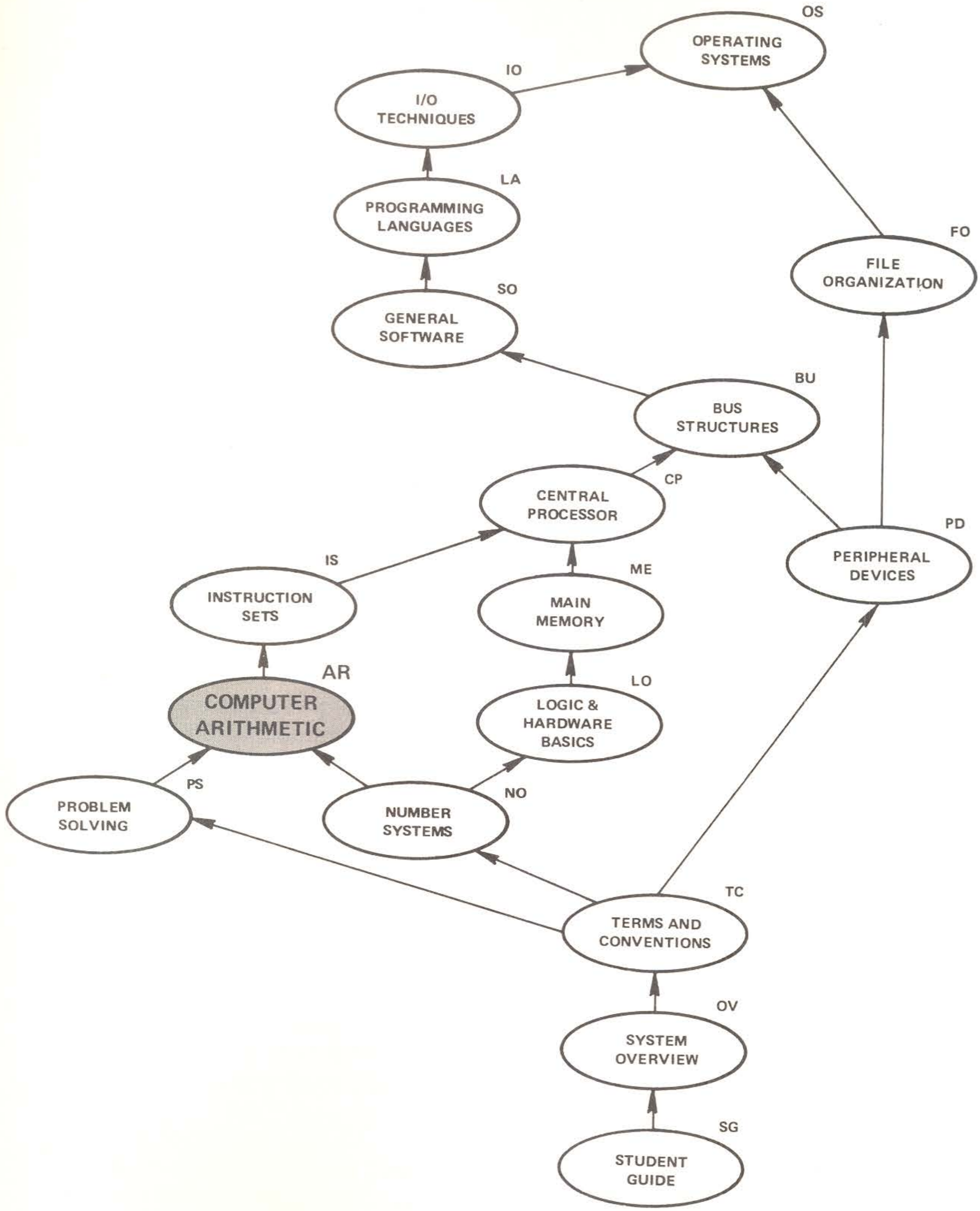
INTRODUCTION TO MINICOMPUTERS

Computer Arithmetic

Student Workbook

Audio-Visual Course by Digital Equipment Corporation

COURSE MAP



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Computer Arithmetic

Introduction

This module is about how computers do arithmetic. You have already seen that computers do all their work in binary, and that programmers often work in octal because it is easier to convert from octal to binary than from decimal to binary. However, we all grew up learning to do arithmetic in decimal. This module relates octal and binary arithmetic to the corresponding operations in decimal to make them easy for you to understand.

All arithmetic problems can be reduced to four basic operations: addition, subtraction, multiplication, and division. To simplify their design, many digital computers reduce these four basic operations to just one operation: addition. Subtraction is performed by a process called *complementary addition.* Multiplication is performed by repeated additions. Division is performed by repeated subtractions which, in turn, are performed by complementary addition.

Therefore, it is useful to know how to perform addition, subtraction, and complementary addition in the number systems commonly used when working with minicomputers. This module explains how to perform these three operations in octal and binary by relating them to the equivalent operations in decimal. The module then discusses ways of expressing positive and negative numbers in binary.

This module is totally in workbook form, without audio-visual material. Work through each lesson at your own pace and discuss any problems with your course manager.

Addition

OBJECTIVE

Given pairs of octal numbers and pairs of binary numbers, be able to add each pair.

SAMPLE TEST ITEMS

Add:

1. $$\begin{array}{r} 27_8 \\ +41_8 \\ \hline \end{array}$$

2. $$\begin{array}{r} 364_8 \\ +275_8 \\ \hline \end{array}$$

3. $$\begin{array}{r} 101101_2 \\ +\ 10111_2 \\ \hline \end{array}$$

4. $$\begin{array}{r} 11101110_2 \\ +10110100_2 \\ \hline \end{array}$$

Addition

In all number systems, addition is performed in the same way. A set of addition *facts,* such as 2 + 3 = 5, is derived for every possible combination of digits in the number system. These facts are then applied with a method for *carrying* digits to columns with higher place values when necessary.

Table 1 shows the set of addition facts for the octal number system. If you are as familiar with them as you are with the decimal facts, you can add numbers in this base directly. That is, you can "think" octal. Most people, however, find their decimal addition facts so deeply ingrained that it is almost impossible for them to think of 13 when they see 6 + 5. To make the task easier, this lesson will show you how to add in octal and binary while still "thinking" in decimal. The lesson begins by deriving a generalized system for adding numbers using base 10, and then applies this system to addition in bases 8 and 2.

Table 1
Octal Addition Facts

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	10
2	2	3	4	5	6	7	10	11
3	3	4	5	6	7	10	11	12
4	4	5	6	7	10	11	12	13
5	5	6	7	10	11	12	13	14
6	6	7	10	11	12	13	14	15
7	7	10	11	12	13	14	15	16

Decimal Addition

Single-Digit Problems – What do you do when you add numbers? Let's answer this by looking at the simplest type of addition problem:

Find the sum of two and four.

We usually begin by writing the numbers to be added in columns:

$$\begin{array}{r} 2 \\ +\,4 \\ \hline \end{array}$$

Then we add the digits by referring to a table of addition facts, if necessary, and write the sum in the correct column:

$$\begin{array}{r} 2 \\ +\,4 \\ \hline 6 \end{array}$$

This example, 2 + 4, involves only one column. It does not require *carrying* because the sum of the column (6) is *less* than the base of the number system in which we are working (10).

Figure 1 is a flowchart of the addition process for single-digit problems that do not require carrying. To verify this flowchart, follow it through for another simple problem:

Find the sum of one and six.

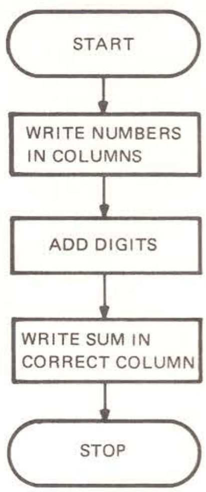


Figure 1 Addition for Single-Digit Problems Without Carrying

Multidigit Problems – When addition problems involve numbers with more than one digit, the process has a small twist. Here is a multidigit problem that does not require carrying:

Find the sum of 23 and 62.

Again, the numbers are written in columns:

$$\begin{array}{rcc} & 2 & 3 \\ + & 6 & 2 \\ \hline \end{array}$$

Then we add the digits beginning with the rightmost column:

$$\begin{array}{rc|c|} & 2 & 3 \\ + & 6 & 2 \\ \hline & & 5 \end{array}$$

Moving from right to left, we add each column until all columns have been added:

$$\begin{array}{r|c|c} & 2 & 3 \\ + & 6 & 2 \\ \hline & 8 & 5 \end{array}$$

Figure 2 is a flowchart of the addition process for multidigit problems that do not require carrying. Compare this flowchart to the one in Figure 1. The new blocks that were added have bold outlines. Verify the process by following the flowchart through for these problems:

1. Find the sum of 41 and 33. (74)
2. Find the sum of 18 and 50. (68)

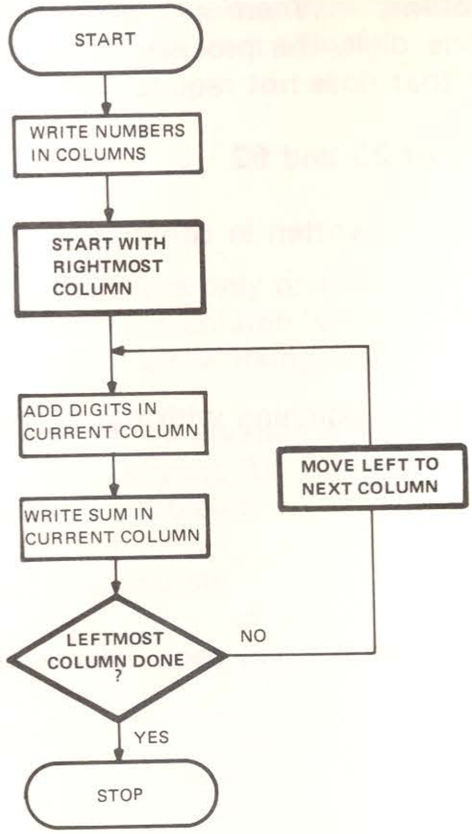


Figure 2 Addition for Multidigit Problems Without Carrying

Carrying – Suppose the sum of a column is *greater* than the base of the number system:

$$\begin{array}{r} 4\,\boxed{7} \\ +\,3\,\boxed{5} \\ \hline \end{array}$$

In this problem, 7 + 5 = 12, and 12 is greater than the base of the number system, 10. We must therefore *carry* a digit to the column with the next higher place value:

$$\begin{array}{r} 4\,\boxed{7} \\ +\,3\,\boxed{5} \\ \hline \boxed{2} \end{array} \quad \text{with a carry of 1}$$

The carry digit is added to the digits in the next column to the left:

$$\begin{array}{r} \boxed{4}\,7 \\ +\,\boxed{3}\,5 \\ \hline \boxed{8}\,2 \end{array}$$

If the sum of the last column is greater than the base of the number system, the carry digit is simply written to the left of the sum:

$$\begin{array}{r} 9\,4 \\ +\;\;7\,2 \\ \hline \boxed{1}\,6\,6 \end{array}$$

carry digit for last column

Let's examine the process of determining the value of the carry digit in more detail. Since you are so familiar with base 10 arithmetic, this process may not be immediately obvious. How do you determine the carry digit for the rightmost column in the following problem?

$$\begin{array}{r} 3\,4 \\ 2\,6 \\ 1\,8 \\ +\,1\,7 \\ \hline \end{array}$$

First, we add 4 + 6 + 8 + 7 = 25. This result, 25, is *divided* by the base of the number system, 10:

$$\begin{array}{r l} 2 & \leftarrow \text{Quotient} \\ 10\overline{)\,25} & \\ 20 & \\ \hline 5 & \leftarrow \text{Remainder} \end{array}$$

The *remainder* of this division, 5, is the digit written in the current column. The *quotient*, 2, is the digit carried to the next column.

This division is the key to performing addition in number systems other than base 10 while still "thinking" in decimal. Figure 3 expands the flowchart shown in Figure 2 to include the carrying process. Again, the new blocks have bold outlines. Note that this flowchart does *not* need a separate loop for the special case when the sum is less than the base. When this happens, the division will yield a remainder equal to the sum and a quotient of 0:

$$\begin{array}{r} 2\boxed{3} \\ +\,4\boxed{1} \\ \hline \boxed{4} \end{array} \qquad\qquad \begin{array}{rl} 0 & \longleftarrow \text{Quotient} \\ 10\overline{)\,4} & \\ 0 & \\ \hline 4 & \longleftarrow \text{Remainder} \end{array}$$

Carrying 0 is the same as not carrying anything at all.

Verify the process in this flowchart by following it through for these problems:

1. Add 28 and 44. (72)
2. Find the sum of 126 and 95. (221)
3. Add 36, 91, and 77. (204)

Octal Addition

Using the flowchart in Figure 3, you are now ready to begin addition in octal. We will begin with two very simple problems to walk you through the process and then give you problems to try on your own.

Referring to the octal addition facts given in Table 1, you see that:

$$5_8 + 7_8 = 14_8$$

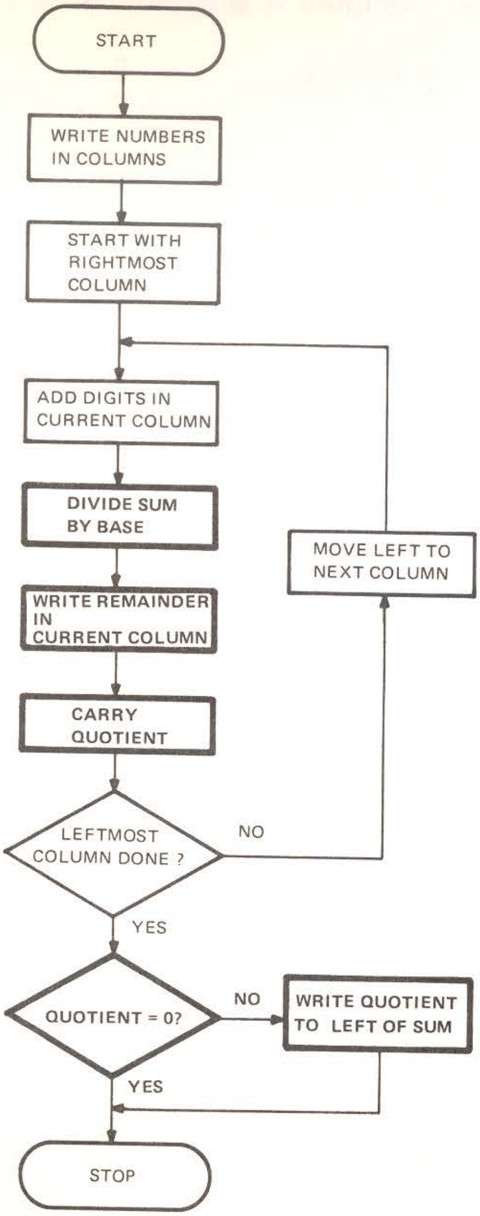


Figure 3 Addition for Multidigit Problems With Carrying

Here is how the process in Figure 3 is applied to this problem:

1. Write numbers in columns.

$$\begin{array}{r} 5 \\ +\,7 \\ \hline \end{array}$$

2. Start with rightmost column.

$$\begin{array}{r} \boxed{5} \\ +\,\boxed{7} \\ \hline \end{array}$$

3. Add digits in current column. (THINK $5_{10} + 7_{10} = 12_{10}$)

4. Divide sum by base.

$$\begin{array}{rl} 1 & \leftarrow \text{Quotient} \\ 8\,\overline{)\,12} & \\ \underline{8} & \\ 4 & \leftarrow \text{Remainder} \end{array}$$

5. Write remainder in current column.

$$\begin{array}{r} \boxed{5} \\ +\,\boxed{7} \\ \hline \boxed{4} \end{array}$$ with a carry of 1

6. Carry quotient. CARRY 1

7. Leftmost column done? YES

8. Quotient = 0? NO

9. Write quotient to left of sum.

$$\begin{array}{rl} 5 & \\ +\;\;7 & \\ \hline \boxed{1}4 & \\ \uparrow\quad & \text{Quotient} \end{array}$$

10. STOP

Note that both the addition in step 3 and the division in step 4 were done in *decimal,* not in octal. Yet the result in step 9 appears in octal by following this process.

Here is a more complex problem:

$64_8 + 75_8 =$

1. Write numbers in columns.

$$\begin{array}{r} 6\,4 \\ +\,7\,5 \\ \hline \end{array}$$

2. Start with rightmost column.

$$\begin{array}{r} 6\,\boxed{4} \\ +\,7\,\boxed{5} \\ \hline \end{array}$$

3. Add digits in current column. (THINK $4_{10} + 5_{10} = 9_{10}$)

4. Divide sum by base.

$$\begin{array}{rl} 1 & \leftarrow \text{Quotient} \\ 8\,\overline{)\,9} & \\ \underline{8} & \\ 1 & \leftarrow \text{Remainder} \end{array}$$

5. Write remainder in current column.

$$\begin{array}{r} 6\,\boxed{4} \\ +\,7\,\boxed{5} \\ \hline \boxed{1} \end{array}$$ with a carry of 1

6. Carry quotient. CARRY 1

7. Leftmost column done? NO

8. Move left to next column.

$$\begin{array}{r} \boxed{6}\,4 \\ +\,\boxed{7}\,5 \\ \hline 1 \end{array}$$ with a carry of 1

9. Add digits in current column. (THINK $6_{10} + 7_{10} + 1_{10} = 14_{10}$)

10. Divide sum by base.

$$\begin{array}{r} 1 \\ 8\,\overline{)\,14} \\ \underline{8} \\ 6 \end{array}$$

11. Write remainder in current column.

$$\begin{array}{r} \boxed{6}\,4 \\ +\,\boxed{7}\,5 \\ \hline \boxed{6}\,1 \end{array}$$

12. Carry quotient. CARRY 1

13. Leftmost column done? YES

14. Quotient = 0? NO

15. Write quotient to left of sum.

$$\begin{array}{r} 64 \\ +\ 75 \\ \hline 161 \end{array}$$

16. STOP

On the next page are nine problems in octal addition. Work these problems and check your answers with the solutions that follow.

EXERCISES

Octal Addition

1. $\begin{array}{r} 21 \\ +55 \\ \hline \end{array}$

2. $\begin{array}{r} 60 \\ +40 \\ \hline \end{array}$

3. $\begin{array}{r} 71 \\ +37 \\ \hline \end{array}$

4. $\begin{array}{r} 670 \\ +415 \\ \hline \end{array}$

5. $\begin{array}{r} 222 \\ +367 \\ \hline \end{array}$

6. $\begin{array}{r} 461 \\ +317 \\ \hline \end{array}$

7. $\begin{array}{r} 2173 \\ +4065 \\ \hline \end{array}$

8. $\begin{array}{r} 1015 \\ +2066 \\ \hline \end{array}$

9. $\begin{array}{r} 5526 \\ +6374 \\ \hline \end{array}$

SOLUTIONS

Octal Addition

1. $\begin{array}{r} 21 \\ +55 \\ \hline 76 \end{array}$

2. $\begin{array}{r} 60 \\ +\ 40 \\ \hline 120 \end{array}$

3. $\begin{array}{r} 71 \\ +\ 37 \\ \hline 130 \end{array}$

4. $\begin{array}{r} 670 \\ +\ 415 \\ \hline 1305 \end{array}$

5. $\begin{array}{r} 222 \\ +367 \\ \hline 611 \end{array}$

6. $\begin{array}{r} 461 \\ +\ 317 \\ \hline 1000 \end{array}$

7. $\begin{array}{r} 2173 \\ +4065 \\ \hline 6260 \end{array}$

8. $\begin{array}{r} 1015 \\ +2066 \\ \hline 3103 \end{array}$

9. $\begin{array}{r} 5526 \\ +\ 6374 \\ \hline 14122 \end{array}$

> If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the section, "Binary Addition."

Binary Addition

The flowchart in Figure 3 can be applied to binary addition as well as octal and decimal. To verify this, we will first find the sum of $1_2 + 1_2$ from a table of binary addition facts. Then we will find the same sum by the process shown in the flowchart.

Table 2 shows the set of addition facts for the binary number system. From this table, you can see that

$$1_2 + 1_2 = 10_2$$

Table 2 Binary Addition Facts

+	0	1
0	0	1
1	1	10

Here is how the process in Figure 3 is applied to this problem:

1. Write numbers in columns.
$$\begin{array}{r} 1 \\ +\,1 \\ \hline \end{array}$$

2. Start with rightmost column.
$$\begin{array}{r} \boxed{1} \\ +\,\boxed{1} \\ \hline \end{array}$$

3. Add digits in current column. (THINK $1_{10} + 1_{10} = 2_{10}$)

4. Divide sum by base.
$$\begin{array}{rl} 1 & \longleftarrow \text{Quotient} \\ 2\,\overline{)\,2} & \\ 2 & \\ \hline 0 & \longleftarrow \text{Remainder} \end{array}$$

5. Write remainder in current column.
$$\begin{array}{r} \boxed{1} \\ +\,\boxed{1} \\ \hline \boxed{0} \end{array}$$ with a carry of 1

6. Carry quotient. CARRY 1

7. Leftmost column done? YES

8. Quotient = 0? NO

9. Write quotient to left of sum.

$$\begin{array}{r} 1 \\ +\ 1 \\ \hline \boxed{1}0 \end{array}$$

Quotient

10. STOP

Again, the addition in step 3 and the division in step 4 are done in decimal, while the result in step 9 appears in binary. The flowchart process allows you to add in any base less than 10 while still "thinking" in decimal.

Here is another problem:

$111_2 + 110_2 =$

Try to work this problem on your own. Then check your work with the solution on the next page.

1. Write numbers in columns.

$$\begin{array}{r} 1\ 1\ 1 \\ +\ 1\ 1\ 0 \\ \hline \end{array}$$

2. Start with rightmost column.

$$\begin{array}{r} 1\ 1\ \boxed{1} \\ +\ 1\ 1\ \boxed{0} \\ \hline \end{array}$$

3. Add digits in current column. (THINK $1_{10} + 0_{10} = 1_{10}$)

4. Divide sum by base.

$$\begin{array}{rl} 0 & \leftarrow \text{Quotient} \\ 2\overline{)\ 1} & \\ 0 & \\ \hline 1 & \leftarrow \text{Remainder} \end{array}$$

5. Write remainder in current column.

$$\begin{array}{r} 1\ 1\ \boxed{1} \\ +\ 1\ 1\ \boxed{0} \\ \hline \boxed{1} \end{array} \quad \text{with a carry of 0}$$

6. Carry quotient. CARRY 0

7. Leftmost column done? NO

8. Move left to next column.

$$\begin{array}{r} 1\ \boxed{1}\ 1 \\ +\ 1\ \boxed{1}\ 0 \\ \hline 1 \end{array} \quad \text{with a carry of 0}$$

9. Add digits in current column. (THINK $1_{10} + 1_{10} + 0_{10} = 2_{10}$)

10. Divide sum by base.

$$\begin{array}{r} 1 \\ 2\overline{)\ 2} \\ 2 \\ \hline 0 \end{array}$$

11. Write remainder in current column.

$$\begin{array}{r} 1\ \boxed{1}\ 1 \\ +\ 1\ \boxed{1}\ 0 \\ \hline \boxed{0}\ 1 \end{array}$$

12. Carry quotient. CARRY 1

13. Leftmost column done? NO

14. Move left to next column.

$$\begin{array}{r} \boxed{1}\;1\;1 \\ +\;\boxed{1}\;1\;0 \\ \hline 0\;1 \end{array}$$

15. Add digits in current column. (THINK $1_{10} + 1_{10} + 1_{10} = 3_{10}$)

16. Divide sum by base.

$$\begin{array}{r} 1 \\ 2\,\overline{)\;3} \\ \underline{2} \\ 1 \end{array}$$

17. Write remainder in current column.

$$\begin{array}{r} \boxed{1}\;1\;1 \\ +\;\boxed{1}\;1\;0 \\ \hline \boxed{1}\;0\;1 \end{array}$$

18. Carry quotient. CARRY 1

19. Leftmost column done? YES

20. Quotient = 0? NO

21. Write quotient to left of sum.

$$\begin{array}{r} 1\;1\;1 \\ +\;1\;1\;0 \\ \hline 1\;1\;0\;1 \end{array}$$

22. STOP

This example was rather long. To verify that the answer is correct, we can convert 111_2, 110_2, and 1101_2 to decimal as a check.

$$
\begin{aligned}
111_2 &= 1 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\
&= 1 \times 4 + 1 \times 2 + 1 \times 1 \\
&= 4 + 2 + 1 \qquad = 7_{10}
\end{aligned}
$$

$$
\begin{aligned}
110_2 &= 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\
&= 1 \times 4 + 1 \times 2 + 0 \times 1 \\
&= 4 + 2 + 0 \qquad = 6_{10}
\end{aligned}
$$

$$
\begin{aligned}
1101_2 &= 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\
&= 1 \times 8 + 1 \times 4 + 0 \times 2 + 1 \times 1 \\
&= 8 + 4 + 0 + 1 \qquad =13_{10}
\end{aligned}
$$

Since $7_{10} + 6_{10} = 13_{10}$, we can be sure that $111_2 + 110_2 = 1101_2$. Therefore, the process in Figure 3 is valid for binary addition.

> The next page contains six problems in binary addition. Work these problems and check your answers with the solutions that follow.

EXERCISES

Binary Addition

1. $$\begin{array}{r} 1010 \\ +1110 \\ \hline \end{array}$$

2. $$\begin{array}{r} 1101 \\ +\ \ 111 \\ \hline \end{array}$$

3. $$\begin{array}{r} 11010 \\ +\ \ 1110 \\ \hline \end{array}$$

4. $$\begin{array}{r} 10101 \\ +10101 \\ \hline \end{array}$$

5. $$\begin{array}{r} 111011 \\ +101101 \\ \hline \end{array}$$

6. $$\begin{array}{r} 110110 \\ +101110 \\ \hline \end{array}$$

SOLUTIONS

Binary Addition

1.
$$\begin{array}{r} 1010 \\ +\ 1110 \\ \hline 11000 \end{array}$$

2.
$$\begin{array}{r} 1101 \\ +\ 111 \\ \hline 10100 \end{array}$$

3.
$$\begin{array}{r} 11010 \\ +\ 1110 \\ \hline 101000 \end{array}$$

4.
$$\begin{array}{r} 10101 \\ +\ 10101 \\ \hline 101010 \end{array}$$

5.
$$\begin{array}{r} 111011 \\ +\ 101101 \\ \hline 1101000 \end{array}$$

6.
$$\begin{array}{r} 110110 \\ +\ 101110 \\ \hline 1100100 \end{array}$$

> If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the next lesson, "Direct Subtraction."

Direct Subtraction

OBJECTIVE

Given pairs of octal numbers and pairs of binary numbers, be able to subtract each pair using the direct subtraction method.

SAMPLE TEST ITEMS

Subtract directly:

1. $\begin{array}{r} 34_8 \\ -15_8 \\ \hline \end{array}$

2. $\begin{array}{r} 326_8 \\ -244_8 \\ \hline \end{array}$

3. $\begin{array}{r} 110110_2 \\ -\ 11101_2 \\ \hline \end{array}$

4. $\begin{array}{r} 10010000_2 \\ -\ 1101111_2 \\ \hline \end{array}$

Subtraction

Most minicomputers do not subtract directly. Instead, they use a process called complementary addition. Some minicomputers, however, do subtract directly, and direct subtraction is easier for you to use than complementary addition.

This lesson teaches direct subtraction through the same approach used in the previous lesson. That is, it develops a generalized system for subtracting numbers using base 10 and then applies this system to subtraction in bases 8 and 2. This approach will allow you to subtract in any base less than 10 while still "thinking" in decimal.

Before we begin, let's review the terms used in discussing subtraction. These are:

- Minuend – the number from which the subtraction will be made
- Subtrahend – the number to be subtracted
- Difference – the answer in a subtraction problem

These quantities are identified in the following example:

$$\begin{array}{rl} 245 & \longleftarrow \text{Minuend} \\ -173 & \longleftarrow \text{Subtrahend} \\ \hline 72 & \longleftarrow \text{Difference} \end{array}$$

Decimal Subtraction

Multidigit Problems – The steps in subtraction are similar to those in addition. They are so similar, in fact, that to convert the flowchart for addition of multidigit problems in Figure 2 to subtraction, we need only to change two words: add to subtract and sum to difference. Figure 4 shows these changes in the boxes that have bold outlines. This process is valid for problems of any size, but it does not allow for problems that require borrowing.

Verify the process by following it through for the problems below.

1. Find the difference between 57 and 33. (24)

2. What result is obtained when 25 is subtracted from 86? (61)

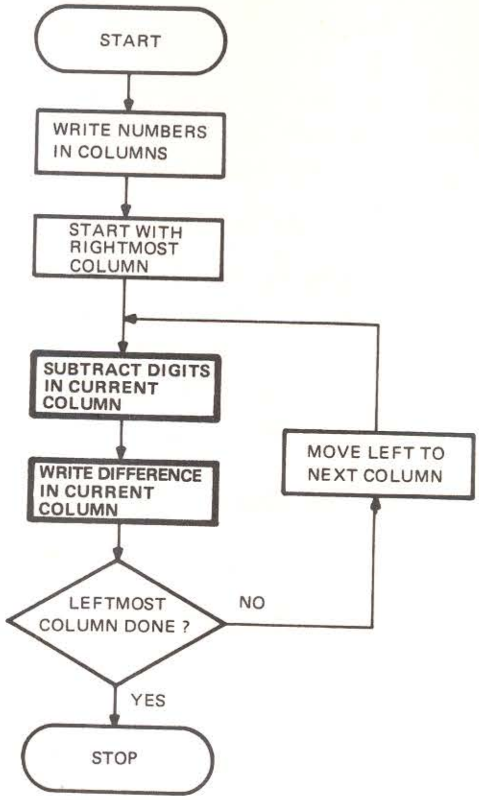


Figure 4 Subtraction for Multidigit Problems Without Borrowing

Simple Borrowing – If a digit in the subtrahend is *greater* than the corresponding digit in the minuend, you must *borrow* before you can subtract.

$$\begin{array}{rl} 8\,\boxed{2} & \longleftarrow \text{Minuend} \\ -\,2\,\boxed{6} & \longleftarrow \text{Subtrahend} \\ \hline \end{array}$$

In the problem above, 6 is greater than 2. To accomplish the subtraction, the value of the base, 10, is borrowed from the column with the next higher place value. This value is added to the digit in the minuend. The digit in the subtrahend can then be subtracted from their sum:

$$\begin{array}{r|c|} 7 & 12 \\ \cancel{8} & \cancel{2} \\ -\,2 & 6 \\ \hline & 6 \end{array}$$

Borrowing decreases the value of the next higher column by one. This must be remembered when you move on to that column:

$$
\begin{array}{rcc}
 & 7 & \\
 & \cancel{8} & 2 \\
- & 2 & 6 \\
\hline
 & 5 & 6
\end{array}
$$

The key to this process is that *the value borrowed is the value of the base in which you are working.* In decimal, you borrow 10. In octal, you borrow 8. In binary, you borrow 2. The borrowed value is added to the digit in the minuend, and the subtraction is done as if you are working in base 10. This process allows you to subtract in any base less than 10 while still "thinking" in decimal.

Figure 5 is a flowchart of the process just described. The blocks added to Figure 4 have bold outlines. Remember that the symbol > in the bold decision block means "greater than." Thus, this block asks "Is the subtrahend greater than the minuend?"

Verify the process in this flowchart by following it through for these problems:

1. 46 − 38 = (8)
2. 241 − 87 = (154)
3. 526 − 398 = (128)

Complex Borrowing – Look at the problem below:

$$
\begin{array}{rccc}
 & 5 & 0 & 4 \\
- & 2 & 2 & 7 \\
\hline
\end{array}
$$

We need to borrow to subtract 7 from 4 in the first column, but the next column is a zero.

$$
\begin{array}{rccc}
 & & ? & ? \\
 & 5 & \cancel{0} & \cancel{4} \\
- & 2 & 2 & 7 \\
\hline
\end{array}
$$

When this happens, we must borrow from the next *non-zero* column. In this case, the 5 becomes a 4 and the 0 becomes a 10:

$$
\begin{array}{rccc}
 & 4 & 10 & ? \\
 & \cancel{5} & \cancel{0} & \cancel{4} \\
- & 2 & 2 & 7 \\
\hline
\end{array}
$$

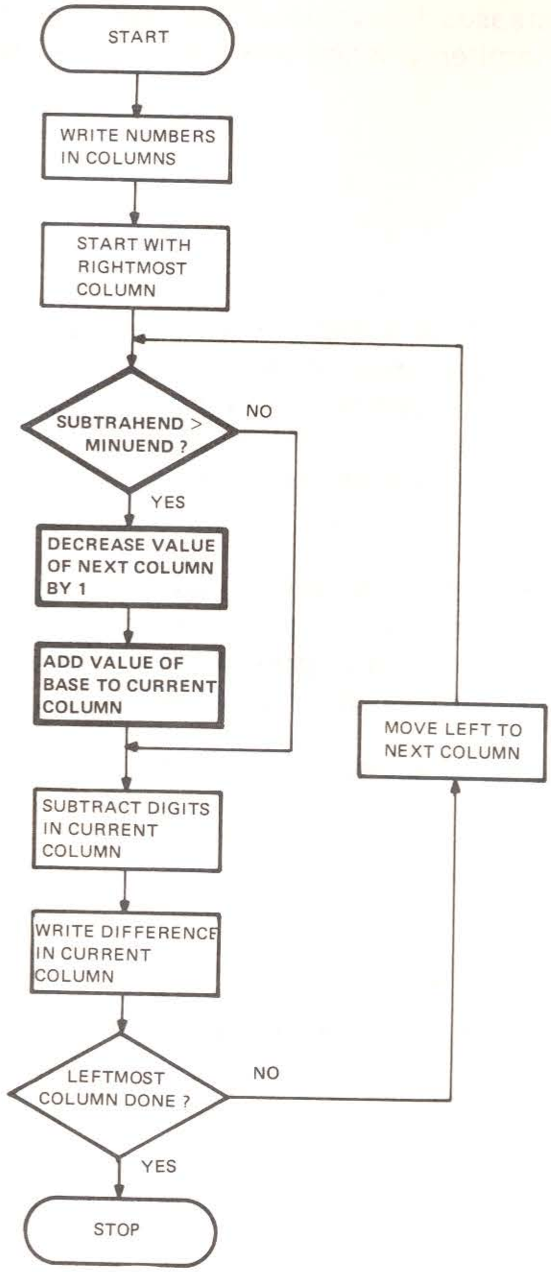


Figure 5 Subtraction for Multidigit Problems with Simple Borrowing

We can then borrow from the second column, decreasing it by 1 and adding the value of the base to the current column:

$$\begin{array}{rrr} & 9 & \\ 4 & \cancel{10} & 14 \\ \cancel{5} & \cancel{0} & \cancel{4} \\ -2 & 2 & 7 \\ \hline \end{array}$$

The subtraction can now be carried out as follows:

$$\begin{array}{rrr} 4 & 9 & 14 \\ \cancel{5} & \cancel{0} & \cancel{4} \\ -2 & 2 & 7 \\ \hline 2 & 7 & 7 \end{array}$$

The essential elements of the process are:

- borrowing is done from the next *non-zero* column, and
- all intervening columns (between the current column and the next non-zero column) are set *to the value of the base minus 1.* This technique is shown in the bold boxes in Figure 6.

Verify this process by following it through for the following problems.

1. 204 - 86 = (118)
2. 5002 - 117 = (4885)
3. 7000 - 983 = (6017)

Octal Subtraction

Now let's perform subtraction in octal. We will use the flowchart in Figure 6 to work through two sample problems and then supply problems for you to work on your own.

Here is the first problem:

$$14_8 - 6_8 =$$

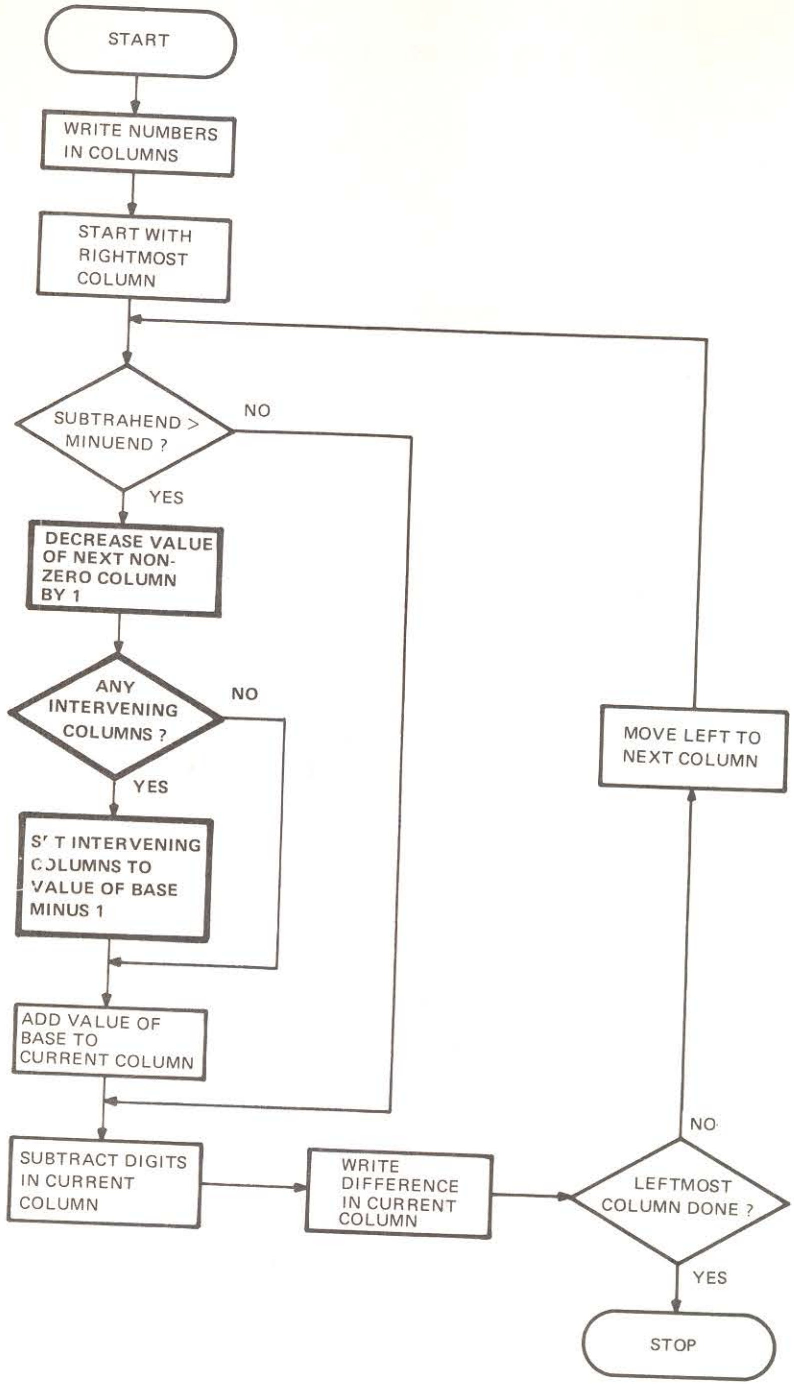


Figure 6 Subtraction for Multidigit Problems with Complex Borrowing

In this problem it is simple enough to convert the minuend and subtrahend to base 10, subtract, and convert the difference back to base 8:

$$\begin{array}{lrcr} \text{Minuend:} & 14_8 & = & 12_{10} \\ \text{Subtrahend:} & -\ 6_8 & = & -6_{10} \\ \hline \text{Difference:} & & & 6_{10} = 6_8 \end{array}$$

To do all the thinking in decimal, the process in Figure 6 is applied like this:

1. Write numbers in columns.

$$\begin{array}{r} 1\ 4 \\ -\ \ 6 \\ \hline \end{array}$$

2. Start with rightmost column.

$$\begin{array}{rr} 1 & \boxed{4} \\ - & \boxed{6} \\ \hline \end{array}$$

3. Subtrahend > minuend? YES

4 Decrease value of next nonzero column by 1.

$$\begin{array}{rr} 0 & \\ \cancel{1} & \boxed{4} \\ - & \boxed{6} \\ \hline \end{array}$$

5. Any intervening digits? NO

6. Add value of base to current column.

$$\begin{array}{rr} 0 & 12 \\ \cancel{1} & \cancel{4} \\ - & 6 \\ \hline \end{array}$$

(THINK $8_{10} + 4_{10} = 12_{10}$)

7. Subtract digits in current column. (THINK $12_{10} - 6_{10} = 6_{10}$)

8. Write difference in current column.

$$\begin{array}{rr} 0 & \\ \cancel{1} & \boxed{4} \\ - & \boxed{6} \\ \hline & \boxed{6} \end{array}$$

9. Leftmost column done? NO

10. Move left to next column.

$$\begin{array}{rr} & \boxed{0} \\ & \cancel{1}4 \\ - & \boxed{}6 \\ \hline & 6 \end{array}$$

11. Subtrahend > minuend? NO

12. Subtract digits in current column. (THINK $0_{10} - 0_{10} = 0_{10}$)

13. Write difference in current column.

$$\begin{array}{rr} & \boxed{1}4 \\ - & 6 \\ \hline & \boxed{0}6 \end{array}$$

14. Leftmost column done? YES

15. STOP

Note that the value of the base, 8, is *borrowed* in step 4. This value is added to the digit in the minuend, 4, to get 12_{10} (step 6). This addition and the subsequent subtraction in step 7 are both done in decimal, not octal. The difference written in step 8 appears in octal by following the flowchart procedure.

One further point: when you are working in the leftmost column and the difference is 0 (steps 10 through 12 above), you need not *write* the difference as we did in step 13. That is, the answer to $14_8 - 6_8$ is usually written as 6 rather than 06, but both answers are correct.

Here is another problem:

$403_8 - 165_8 =$

Try to work this problem on your own. Then check your work with the solution on the next page.

1. Write numbers in columns.

$$\begin{array}{rrrl} 4 & 0 & 3 & \leftarrow \text{Minuend} \\ -1 & 6 & 5 & \leftarrow \text{Subtrahend} \\ \hline \end{array}$$

2. Start with rightmost column.

$$\begin{array}{rrr} 4 & 0 & \boxed{3} \\ -1 & 6 & \boxed{5} \\ \hline \end{array}$$

3. Subtrahend > minuend?

YES

4. Decrease value of next non-zero column by 1.

$$\begin{array}{rrr} 3 & & \\ \cancel{4} & 0 & \boxed{3} \\ -1 & 6 & \boxed{5} \\ \hline \end{array}$$

5. Any intervening columns?

YES

6. Set intervening columns to value of base minus 1.

$$\begin{array}{rrr} 3 & 7 & \\ \cancel{4} & \cancel{0} & \boxed{3} \\ -1 & 6 & \boxed{5} \\ \hline \end{array}$$

7. Add value of base to current column.

$$\begin{array}{rrr} 3 & 7 & \boxed{11} \\ \cancel{4} & \cancel{0} & \cancel{3} \\ -1 & 6 & \boxed{5} \\ \hline \end{array}$$

(THINK $8_{10} + 3_{10} = 11_{10}$)

8. Subtract digits in current column.

(THINK $11_{10} - 5_{10} = 6_{10}$)

9. Write difference in current column.

$$\begin{array}{rrr} 3 & 7 & \\ \cancel{4} & \cancel{0} & \boxed{3} \\ -1 & 6 & \boxed{5} \\ \hline & & \boxed{6} \end{array}$$

10. Leftmost column done?

NO

11. Move left to next column.

$$\begin{array}{rccc} & 3 & \boxed{7} & \\ & \cancel{4} & \boxed{\cancel{0}} & 3 \\ - & 1 & \boxed{6} & 5 \\ \hline & & & 6 \end{array}$$

12. Subtrahend > minuend? NO

13. Subtract digits in current column. (THINK $7_{10} - 6_{10} = 1_{10}$)

14. Write difference in current column.

$$\begin{array}{rccc} & 3 & & \\ & \cancel{4} & \boxed{0} & 3 \\ - & 1 & \boxed{6} & 5 \\ \hline & & \boxed{1} & 6 \end{array}$$

15. Leftmost column done? NO

16. Move left to next column.

$$\begin{array}{rccc} & \boxed{3} & & \\ & \boxed{\cancel{4}} & 0 & 3 \\ - & \boxed{1} & 6 & 5 \\ \hline & & 1 & 6 \end{array}$$

17. Subtrahend > minuend? NO

18. Subtract digits in current column. (THINK $3_{10} - 1_{10} = 2_{10}$)

19. Write difference in current column.

$$\begin{array}{rccc} & \boxed{4} & 0 & 3 \\ - & \boxed{1} & 6 & 5 \\ \hline & \boxed{2} & 1 & 6 \end{array}$$

20. Leftmost column done? YES

21. STOP

There are six problems in octal subtraction on the next page. Solve these problems and then check your answers with the solutions that follow.

EXERCISES

Octal Subtraction

1.
$$\begin{array}{r} 43 \\ -26 \\ \hline \end{array}$$

2.
$$\begin{array}{r} 206 \\ -\ \ 35 \\ \hline \end{array}$$

3.
$$\begin{array}{r} 522 \\ -\ \ 34 \\ \hline \end{array}$$

4.
$$\begin{array}{r} 6372 \\ -\ \ 775 \\ \hline \end{array}$$

5.
$$\begin{array}{r} 3045 \\ -1124 \\ \hline \end{array}$$

6.
$$\begin{array}{r} 6342 \\ -1534 \\ \hline \end{array}$$

Octal Subtraction

1.
```
  43
 -26
 ---
  15
```

2.
```
 206
- 35
 ---
 151
```

3.
```
 522
- 34
 ---
 466
```

4.
```
 6372
- 775
 ----
 5375
```

5.
```
 3045
-1124
 ----
 1721
```

6.
```
 6342
-1534
 ----
 4606
```

If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the section, "Binary Subtraction."

Binary Subtraction

Subtraction in binary can be done just like subtraction in octal. The only difference is that the value borrowed is 2 rather than 8. We will begin with a simple example:

$$10_2 - 1_2 =$$

The flowchart in Figure 6 is used as follows:

1. Write numbers in columns.

$$\begin{array}{rr} 1 & 0 \\ - & 1 \\ \hline \end{array}$$

2. Start with rightmost column.

$$\begin{array}{r|c|} 1 & 0 \\ - & 1 \\ \hline \end{array}$$

3. Subtrahend > minuend? YES

4. Decrease value of next non-zero column by 1.

$$\begin{array}{r|c|} 0 & \\ \cancel{1} & 0 \\ - & 1 \\ \hline \end{array}$$

5. Any intervening digits? NO

6. Add value of base to current column.

$$\begin{array}{r|c|} 0 & 2 \\ \cancel{1} & \cancel{0} \\ - & 1 \\ \hline \end{array}$$

(THINK $2_{10} - 0_{10} = 2_{10}$)

7. Subtract digits in current column. (THINK $2_{10} - 1_{10} = 1_{10}$)

8. Write difference in current column.

$$\begin{array}{r|c|} 0 & 2 \\ \cancel{1} & \cancel{0} \\ - & 1 \\ \hline & 1 \end{array}$$

9. Leftmost column done? NO

10. Move left to next column.

$$\begin{array}{rcc} & \boxed{0} & \\ & \cancel{1} & 0 \\ - & \boxed{} & 1 \\ \hline & & 1 \end{array}$$

11. Subtrahend > minuend? NO

12. Subtract digits in current column. (THINK $0_{10} - 0_{10} = 0_{10}$)

13. Write difference in current column.

$$\begin{array}{rcc} & \boxed{0} & \\ & \cancel{1} & 0 \\ - & & 1 \\ \hline & \boxed{0} & 1 \end{array}$$

14. Leftmost column done? YES

15. STOP

Again, you might skip step 13 since 0s are generally not written to the left of numbers. That is, 01 is usually written simply as 1.

Here is another problem

$1001_2 - 10_2 =$

Try to work this problem on your own. Then check your answer by studying the solution on the next page.

1. Write numbers in columns.

$$\begin{array}{rr} & 1\,0\,0\,1 \\ - & 1\,0 \\ \hline \end{array}$$

2. Start with rightmost column.

$$\begin{array}{rr} & 1\,0\,0\boxed{1} \\ - & 1\boxed{0} \\ \hline \end{array}$$

3. Subtrahend > minuend?

NO

4. Subtract digits in current column.

(THINK $1_{10} - 0_{10} = 1_{10}$)

5. Write difference in current column.

$$\begin{array}{rr} & 1\,0\,0\boxed{1} \\ - & 1\boxed{0} \\ \hline & \boxed{1} \end{array}$$

6. Leftmost column done?

NO

7. Move left to next column.

$$\begin{array}{rr} & 1\,0\boxed{0}1 \\ - & \boxed{1}0 \\ \hline & 1 \end{array}$$

8. Subtrahend > minuend?

YES

9. Decrease value of next non-zero column by 1.

$$\begin{array}{rr} & 0 \\ & \cancel{1}\,0\boxed{0}1 \\ - & \boxed{1}0 \\ \hline & 1 \end{array}$$

10. Any intervening columns?

YES

11. Set intervening columns to value of base minus 1.

$$\begin{array}{rr} & 0\,1 \\ & \cancel{1}\,\cancel{0}\boxed{0}1 \\ - & \boxed{1}0 \\ \hline & 1 \end{array}$$

12. Add value of base to current column.

$$\begin{array}{rcccc} & 0 & 1 & \boxed{2} & \\ & \cancel{1} & \cancel{0} & \boxed{\cancel{0}} & 1 \\ - & & & \boxed{1} & 0 \\ \hline \end{array}$$

13. Subtract digits in current column.

(THINK $2_{10} - 1_{10} = 1_{10}$)

14. Write difference in current column.

$$\begin{array}{rcccc} & 0 & 1 & & \\ & \cancel{1} & \cancel{0} & \boxed{0} & 1 \\ - & & & \boxed{1} & 0 \\ \hline & & & \boxed{1} & 1 \end{array}$$

15. Leftmost column done?

NO

16. Move left to next column.

$$\begin{array}{rcccc} & 0 & \boxed{1} & & \\ & \cancel{1} & \boxed{\cancel{0}} & 0 & 1 \\ & & \boxed{} & 1 & 0 \\ \hline & & & 1 & 1 \end{array}$$

17. Subtrahend > minuend?

NO

18. Subtract digits in current column.

(THINK $1_{10} - 0_{10} = 1_{10}$)

19. Write difference in current column.

$$\begin{array}{rcccc} & 0 & & & \\ & \cancel{1} & \boxed{0} & 0 & 1 \\ - & & \boxed{} & 1 & 0 \\ \hline & & \boxed{1} & 1 & 1 \end{array}$$

20. Leftmost column done?

NO

21. Move left to next column.

$$\begin{array}{rcccc} & \boxed{0} & & & \\ & \boxed{\cancel{1}} & 0 & 0 & 1 \\ - & \boxed{} & & 1 & 0 \\ \hline & & 1 & 1 & 1 \end{array}$$

22. Subtrahend > minuend?

NO

23. Subtract digits in current column. (THINK $0_{10} - 0_{10} = 0_{10}$)

24. Write difference in current column.

$$\begin{array}{r|c|lll} & \boxed{1} & 0 & 0 & 1 \\ - & & & 1 & 0 \\ \hline & \boxed{0} & 1 & 1 & 1 \end{array}$$

25. Leftmost column done? YES

26. STOP

To check this answer, you can convert the numbers to decimal:

$$\begin{array}{rcr} 1001_2 & = & 9_{10} \\ -\quad 10_2 & = & -2_{10} \\ \hline 111_2 & = & 7_{10} \end{array}$$

> The next page contains six problems on binary subtraction. Solve these problems and check your answers with the solutions given on the page that follows.

EXERCISES

Binary Subtraction

1. $\begin{array}{r} 111 \\ -101 \\ \hline \end{array}$

2. $\begin{array}{r} 100 \\ -\quad 1 \\ \hline \end{array}$

3. $\begin{array}{r} 1011 \\ -\ 100 \\ \hline \end{array}$

4. $\begin{array}{r} 1011 \\ -1001 \\ \hline \end{array}$

5. $\begin{array}{r} 1001 \\ -\ 101 \\ \hline \end{array}$

6. $\begin{array}{r} 1111 \\ -1110 \\ \hline \end{array}$

SOLUTIONS

Binary Subtraction

1. $\begin{array}{r} 111 \\ -101 \\ \hline 10 \end{array}$

2. $\begin{array}{r} 100 \\ -\quad 1 \\ \hline 11 \end{array}$

3. $\begin{array}{r} 1011 \\ -\ 100 \\ \hline 111 \end{array}$

4. $\begin{array}{r} 1011 \\ -1001 \\ \hline 10 \end{array}$

5. $\begin{array}{r} 1001 \\ -\ 101 \\ \hline 100 \end{array}$

6. $\begin{array}{r} 1111 \\ -1110 \\ \hline 1 \end{array}$

> If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the next lesson, "Complementary Addition."

Complementary Addition

OBJECTIVE

Given pairs of octal numbers and pairs of binary numbers, be able to subtract each pair using the complementary addition method.

SAMPLE TEST ITEMS

Subtract using complementary addition:

1. $\begin{array}{r} 152_8 \\ -\ 74_8 \\ \hline \end{array}$

2. $\begin{array}{r} 212_8 \\ -105_8 \\ \hline \end{array}$

3. $\begin{array}{r} 00110111_2 \\ -00001011_2 \\ \hline \end{array}$

4. $\begin{array}{r} 111010_2 \\ -\ \ 1111_2 \\ \hline \end{array}$

Complements

Complementary addition is a process used by many minicomputers to perform subtraction. This process is used rather than direct subtraction because it allows the use of the same circuitry as regular addition and therefore simplifies the computer's design. The result of a complementary addition operation will *always* be the same as the corresponding direct subtraction operation, so you can check your work in this lesson by using the techniques that you learned in the last lesson.

As you work, keep in mind that minicomputer architecture provides a *fixed* number of bits to represent numeric values. Bits not required to represent the value are shown as leading zeros.

Example:

With a 12 bit used:

$$5_8 = 000\ 000\ 000\ 101$$
$$1763_8 = 001\ 111\ 110011$$

This "fixed format" permits us to perform complementary addition; without it, the technique would not work. We will add leading zeros to some of our examples to illustrate this concept. In the later section on complementary addition you will see how fixed word size combined with the concept of overflow allows us to subtract by adding.

Once again, let's begin by working in decimal. This will allow you to work with the concept of a complement in a familiar framework. The lesson then follows through to explain complementary addition in octal and binary.

Forming Complements

Radix Complements – The complement of a number is formed by subtracting it from a high number. When that "higher number" is a multiple of the base in which you are working, the complement is called a *radix complement.* (The term "radix" is synonymous with "base.")

The next example shows how the radix complement of a number is formed. To form the radix complement of 6_{10}, we subtract it from the smallest power of the base, 10, that is greater than 6.

$$\begin{array}{r} 10_{10} \\ -\ 6_{10} \\ \hline 4_{10} \end{array}$$

Therefore, 4 is the radix complement of 6 *in base 10.* If we were working in base 8, the radix complement of 6_8 would be 2_8:

$$\begin{array}{r} 10_8 \\ -\ 6_8 \\ \hline 2_8 \end{array}$$

To avoid confusion, the word "radix" in "radix complement" is usually replaced by a number indicating the value of the minuend in the above subtractions. Thus, 4 is called the *ten's complement* of 6, while 2 is called the *eight's complement* of 6. When working in binary, we refer to the *two's complement.*

This process can be summarized as follows:

> *The radix complement of a number is formed by subtracting that number from the smallest power of the base that is greater than that number.*

The following examples illustrate how this rule is applied in decimal, octal, and binary.

1. Find the radix complement of 15 in base 10.
   Solution: The smallest power of the base, 10, that is greater than 15 is 100. Therefore, we subtract 15 from 100:

   $$\begin{array}{r} 100 \\ -\ 15 \\ \hline 85 \end{array}$$

   85 is the *ten's* complement of 15.

   If we had specified a 4-digit, base-10 format, the smallest power of base 10 that is greater than 4 digits is 10000:

   $$\begin{array}{r} 10000 \\ -\ 0015 \\ \hline 9985 \end{array}$$

2. What is the *eight's* complement of 463?
Solution: The smallest power of 8 that is greater than 463_8 is 1000_8.

$$\begin{array}{r} 1000_8 \\ -\ 463_8 \\ \hline 315_8 \end{array}$$

315 is the *eight's* complement of 463.

If we had specified a 4-digit, base-8 format, the smallest power of base 8 that is greater than 4 digits is 10000:

$$\begin{array}{r} 10000_8 \\ -\ 0463_8 \\ \hline 7315_8 \end{array}$$

3. Find the radix complement of binary 1011.
Solution: The smallest power of 2 that is greater than 1011_2 is 10000_2.

$$\begin{array}{r} 10000_2 \\ -\ 1011_2 \\ \hline 0101_2 \end{array}$$

101 is the *two's* complement of 1011.

If we had specified a 6-digit, base-8 format, the smallest power of 2 that is greater than 6 digits is 1000000:

$$\begin{array}{r} 1000000 \\ -\ 001011 \\ \hline 110101 \end{array}$$

NOTE
Format size is important.

There are two points that you should notice about finding radix complements from these examples. First, a number plus its complement will always equal a multiple of the base in which you are working. For example, the *ten's* complement of 23 is 77, and $23_{10} + 77_{10} = 100_{10}$:

$$\begin{array}{r} 100_{10} \\ -\ 23_{10} \\ \hline 77_{10} \end{array} \qquad \begin{array}{r} 23_{10} \\ +\ 77_{10} \\ \hline 100_{10} \end{array}$$

Second, two numbers that are complements actually complement *each other.* That is, we saw that 77 is the *ten's* complement of 23, but 23 is also the *ten's* complement of 77.

The next page contains exercises in finding radix complements. Work these problems before you go on and check your work against the solutions that follow.

EXERCISES

Radix Complements

1. Find the radix complement of each of the following numbers in a 4-digit, base-10 format.

   a. 24 b. 76

   c. 533 d. 1075

2. Find the *eight's* complement of each of the following numbers in a 4-digit, base-8 format.

   a. 24 b. 662

   c. 3013 d. 2475

3. Find the radix complement of each of the following binary numbers in a 6-digit, base-2 format.

   a. 110 b. 1010

   c. 10110 d. 110101

SOLUTIONS

Radix Complements

1. *Ten's* complements:

a.
$$\begin{array}{r} 10000_{10} \\ -\ 0024_{10} \\ \hline 9976_{10} \end{array}$$

b.
$$\begin{array}{r} 10000_{10} \\ -\ 0076_{10} \\ \hline 9924_{10} \end{array}$$

c.
$$\begin{array}{r} 10000_{10} \\ -\ 0533_{10} \\ \hline 9467_{10} \end{array}$$

d.
$$\begin{array}{r} 10000_{10} \\ -\ 1075_{10} \\ \hline 8925_{10} \end{array}$$

2. *Eight's* complements:

a.
$$\begin{array}{r} 10000_{8} \\ -\ 0024_{8} \\ \hline 7754_{8} \end{array}$$

b.
$$\begin{array}{r} 10000_{8} \\ -\ 0662_{8} \\ \hline 7116_{8} \end{array}$$

c.
$$\begin{array}{r} 10000_{8} \\ -\ 3013_{8} \\ \hline 4765_{8} \end{array}$$

d.
$$\begin{array}{r} 10000_{8} \\ -\ 2475_{8} \\ \hline 5303_{8} \end{array}$$

3. *Two's* complements:

a.
$$\begin{array}{r} 1000000_{2} \\ -\ 000110_{2} \\ \hline 111010_{2} \end{array}$$

b.
$$\begin{array}{r} 1000000_{2} \\ -\ 001010_{2} \\ \hline 110110_{2} \end{array}$$

c.
$$\begin{array}{r} 1000000_{2} \\ -\ 010110_{2} \\ \hline 101010_{2} \end{array}$$

d.
$$\begin{array}{r} 1000000_{2} \\ -\ 110101_{2} \\ \hline 001011_{2} \end{array}$$

Radix Minus One Complements – The difficulties in finding radix complements are that you must borrow when you do the subtraction and that you must work in different bases. These difficulties can be avoided by finding the *radix minus one* complement. To do this, simply subtract each digit in the given number from the highest digit in the number system. In decimal, the highest digit is 9. In octal it is 7, and in binary it is 1. Here is an example:

Find the radix minus one complement of 63_{10}.

$$\begin{array}{r} 99_{10} \\ -63_{10} \\ \hline 36_{10} \end{array}$$

36_{10} is the radix minus one complement of 63_{10}.

If we work in base 8, the radix minus one complement of 63_8 is 14_8:

$$\begin{array}{r} 77_{8} \\ -63_{8} \\ \hline 14_{8} \end{array}$$

Since no borrowing is required, the subtraction is done as if we were working in decimal. The term "radix minus one" is usually replaced by a number indicating the value of the minuend in the above subtractions. 36 is called the *nine's complement* of 63, while 14 is called the *seven's complement* of 63. In binary, the radix minus one complement is called the *one's complement.*

This process can be summarized as follows:

> *The radix minus one complement of a number is formed by subtracting each digit in the number from the highest digit in that number system.*

It is important to note that *the radix minus one complement is one less than the radix complement.* The former is simply easier to compute than the latter.

The following examples illustrate how to compute radix minus one complements in decimal, octal, and binary.

1. What is the *nine's* complement of 524?
   Solution: Subtract each digit from 9:

$$\begin{array}{r} 999 \\ -524 \\ \hline 475 \end{array}$$

   475 is the *nine's* complement of 524.

2. Find the *seven's* complement of 7136.
   Solution: Subtract each digit from 7:

$$\begin{array}{r} 7777 \\ -7136 \\ \hline 0641 \end{array}$$

   641 is the *seven's* complement of 7136.

3. Find the radix minus one complement of binary 11010.
   Solution: Subtract each digit from 1:

$$\begin{array}{r} 11111 \\ -11010 \\ \hline 00101 \end{array}$$

   101 is the *one's* complement of 11010.

Again, since no borrowing is required, the subtractions are done as if we were working in decimal. If you need to find the radix complement, you can simply add 1 to the radix minus one complement.

Look more carefully at Example 3. Notice the relationship between the number and its one's complement. Can you see a pattern? Here are some examples to help you:

Number:	101	11101	1001010
One's Complement:	010	00010	0110101

The one's complement of a number is formed by reversing the digits, changing the ones to zeros and the zeros to ones.

This complement can thus be formed *without* subtracting. This point is very important in understanding how computers subtract using complementary addition.

> There are three exercises in finding radix minus one complements on the next page. Do these exercises before you continue. Check your answers with the solutions on the page that follows.

EXERCISES

Radix Minus One Complements

1. Find the *nine's* complements of each of the following decimal numbers.

   a. 64 b. 81

   c. 572 d. 8390

2. Find the *seven's* complements of the following numbers.

   a. 53 b. 261

   c. 4007 d. 31462

3. Find the *one's* complements of the numbers below.

   a. 110 b. 10111

   c. 1101100 d. 101000111

SOLUTIONS

Radix Minus One Complements

1. *Nine's* complements:

a. $\begin{array}{r} 99 \\ -64 \\ \hline 35 \end{array}$ b. $\begin{array}{r} 99 \\ -81 \\ \hline 18 \end{array}$

c. $\begin{array}{r} 999 \\ -572 \\ \hline 427 \end{array}$ d. $\begin{array}{r} 9999 \\ -8390 \\ \hline 1609 \end{array}$

2. *Seven's* complements:

a. $\begin{array}{r} 77 \\ -53 \\ \hline 24 \end{array}$ b. $\begin{array}{r} 777 \\ -261 \\ \hline 516 \end{array}$

c. $\begin{array}{r} 7777 \\ -4007 \\ \hline 3770 \end{array}$ d. $\begin{array}{r} 77777 \\ -31462 \\ \hline 46315 \end{array}$

3. *One's* complements (done by reversing digits):

a. 001 b. 01000

c. 0010011 d. 010111000

> If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the section "Decimal Complementary Addition."

Decimal Complementary Addition

Complementary addition allows computers to subtract by adding. It is performed by replacing the subtrahend with its radix complement and then adding. When this is done, two factors must be considered: *overflow* and *unequal numbers of digits* in the subtrahend and minuend. We will look at each factor in turn.

Overflow – Let's see what happens when complementary addition is performed (the following examples are all in decimal):

	Direct Subtraction		**Complementary Addition**
Minuend:	85		85
Subtrahend:	−34	Radix Complement →	+ 66
Difference:	51		151

The answer derived by complementary addition is 100 greater than that derived by direct subtraction. Expressed differently, the answer derived by complementary addition has an extra 1 at the left. Look at another example:

Direct Subtraction		**Complementary Addition**
236		236
−184	Radix Complement →	+ 816
52		1052

Again, the answer derived by complementary addition has an extra 1 at the left.

This extra digit is called an *overflow.* The sum "overflows" into a column with a higher place value than is found in either of the two numbers that were added. This condition is analogous to an automobile odometer that has reached its limit of 99999.9 miles:

9	9	9	9	9	9

When one more tenth of a mile is traveled, the odometer returns to 00000.0:

1	0	0	0	0	0	0

The mileage count overflows the odometer, and the 1 that should appear to the left of the zeros to read 100000.0 is lost.

To remedy the differences between answers derived by direct subtraction and those found by complementary addition, we simply ignore the overflow. Thus, the first problem yields 51:

$$\begin{array}{r} 85 \\ -34 \\ \hline 51 \end{array} \xrightarrow[\text{Complement}]{\text{Radix}} \begin{array}{r} 85 \\ +\ 66 \\ \hline 1\,51 \end{array}$$

51 = 51

The second problem yields 52:

$$\begin{array}{r} 236 \\ -184 \\ \hline 52 \end{array} \xrightarrow[\text{Complement}]{\text{Radix}} \begin{array}{r} 236 \\ +\ 816 \\ \hline +1\,052 \end{array}$$

52 = 052

This basic process is flowcharted in Figure 7.

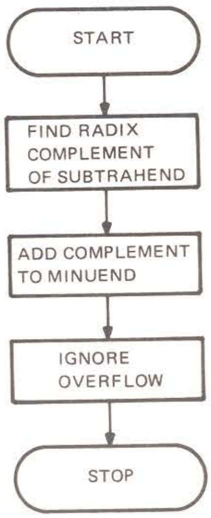


Figure 7 Complementary Addition and Overflow

Use this flowchart to solve the next problem on your own using complementary addition. Check your work with the solution on the next page.

$824_{10} - 777_{10} =$

$$\begin{array}{r}824\\ -777\\ \hline \end{array} \xrightarrow[\text{Complement}]{\text{Ten's}} \begin{array}{r}824\\ +\ 223\\ \hline 1\,047\end{array}$$

Answer = 47

Unequal Digit Problems – There is one other factor to consider when doing complementary addition, as illustrated in the next example:

$$\begin{array}{l}\text{Minuend:}\\ \text{Subtrahend:}\\ \\ \end{array} \qquad \begin{array}{r}392\\ -\ 56\\ \hline 336\end{array} \xrightarrow[\text{Complement}]{\text{Ten's}} \begin{array}{r}392\\ +\ 44\\ \hline +436\end{array}$$

?

The two answers do not correspond as before. This problem occurred because the number of digits in the subtrahend is not equal to the number of digits in the minuend. To correct the problem, we simply need to add zeros to the left of the subtrahend until it has the same number of digits as the minuend:

$$\begin{array}{r}392\\ -\ 56\\ \hline 336\end{array} \xrightarrow[\text{Zeros}]{\text{Add}} -056 \xrightarrow[\text{Complement}]{\text{Ten's}} \begin{array}{r}392\\ +\ 944\\ \hline 1\,336\end{array}$$

336 = 336

Here is another example:

$$\begin{array}{r}2105\\ -\ \ 94\\ \hline 2011\end{array} \xrightarrow[\text{Zeros}]{\text{Add}} -0094 \xrightarrow[\text{Complement}]{\text{Radix}} \begin{array}{r}2105\\ +\ 9906\\ \hline 1\,2011\end{array}$$

The flowchart in Figure 8 summarizes this process.

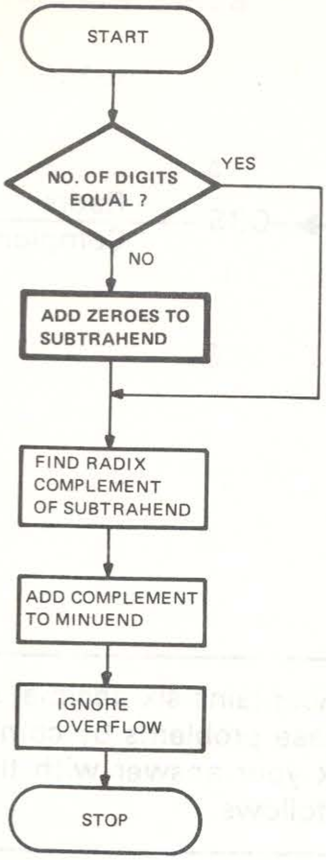


Figure 8 Complementary Addition with Unequal Numbers of Digits in the Minuend and Subtrahend

Use the flowchart to help you solve the next problem, and then compare your answer to that on the next page.

$340_{10} - 85_{10} =$

$$\begin{array}{r}340\\ -\ 85\\ \hline\end{array} \xrightarrow[\text{Zeros}]{\text{Add}} -085 \xrightarrow[\text{Complement}]{\text{Radix}} \begin{array}{r}340\\ +915\\ \hline 1\,255\end{array}$$

Answer = 255

> The next page contains six decimal subtraction problems. Solve these problems by complementary addition and check your answer with the ones given on the page that follows.

EXERCISES

Decimal Complementary Addition

1.	$\begin{array}{r} 76 \\ \underline{-67} \end{array}$	4.	$\begin{array}{r} 6046 \\ \underline{-\ 421} \end{array}$
2.	$\begin{array}{r} 686 \\ \underline{-528} \end{array}$	5.	$\begin{array}{r} 7808 \\ \underline{-\ \ 22} \end{array}$
3.	$\begin{array}{r} 459 \\ \underline{-\ 48} \end{array}$	6.	$\begin{array}{r} 37163 \\ \underline{-15380} \end{array}$

SOLUTIONS

Decimal Complementary Addition

1. $\begin{array}{r} 76 \\ \underline{-67} \end{array} \longrightarrow \begin{array}{r} 76 \\ \underline{+33} \\ 1\,09 \end{array}$

2. $\begin{array}{r} 686 \\ \underline{-528} \end{array} \longrightarrow \begin{array}{r} 686 \\ \underline{+472} \\ 1\,158 \end{array}$

3. $\begin{array}{r} 459 \\ \underline{-\ \ 48} \end{array} \longrightarrow \begin{array}{r} 459 \\ \underline{+952} \\ 1\,411 \end{array}$

4. $\begin{array}{r} 6046 \\ \underline{-\ \ 421} \end{array} \longrightarrow \begin{array}{r} 6046 \\ \underline{+9579} \\ 1\,5625 \end{array}$

5. $\begin{array}{r} 7808 \\ \underline{-\ \ \ \ 22} \end{array} \longrightarrow \begin{array}{r} 7808 \\ \underline{+9978} \\ 1\,7786 \end{array}$

6. $\begin{array}{r} 37163 \\ \underline{-15380} \end{array} \longrightarrow \begin{array}{r} 37163 \\ \underline{+84620} \\ 1\,21783 \end{array}$

> If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the section "Octal Complementary Addition."

Octal Complementary Addition

Complementary addition in octal can follow the same process as decimal complementary addition. However, since it is easier to find radix minus one complements than radix complements in non-decimal bases, a small change is made. Instead of finding the radix complement, we find the radix minus one complement and then add one to get the radix complement. This process is shown in the flowchart in Figure 9.

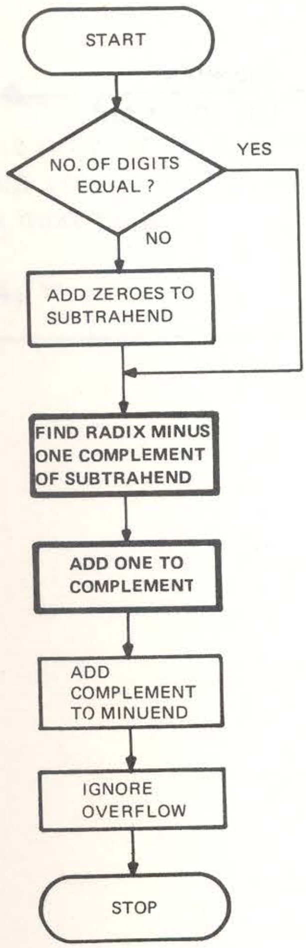


Figure 9 Complementary Addition Using Radix Minus One Complements

Here is an example:

$$\begin{array}{r} 23_8 \\ -15_8 \\ \hline \end{array} \xrightarrow[\text{One Complement}]{\text{Radix Minus}} +62_8 \xrightarrow[\text{One}]{\text{Add}} \begin{array}{r} 23_8 \\ +\ 63_8 \\ \hline 1\,06_8 \end{array}$$

The answer is 6_8. You can check this by subtracting 15_8 from 23_8 by direct subtraction.

The next example has unequal numbers of digits:

$$\begin{array}{r} 444_8 \\ -\ 76_8 \\ \hline \end{array} \xrightarrow[\text{Zeros}]{\text{Add}} -076_8 \xrightarrow[\text{Complement}]{\text{Seven's}} +701_8 \xrightarrow[\text{One}]{\text{Add}} \begin{array}{r} 444_8 \\ +702_8 \\ \hline 1\,346_8 \end{array}$$

Try the next one on your own, using the flowchart in Figure 9 to help you. Check your answers with the solution on the next page.

$642_8 - \ 574_8 =$

$$\begin{array}{r} 642_8 \\ -574_8 \\ \hline \end{array} \longrightarrow +203_8 \longrightarrow \begin{array}{r} 642_8 \\ +\ 204_8 \\ \hline 1\,046_8 \end{array}$$

There are six problems in octal subtraction on the next page. Work these problems using complementary addition and check your work with the solutions that follow.

EXERCISES

Octal Complementary Addition

1. $\begin{array}{r} 44 \\ \underline{-32} \end{array}$

2. $\begin{array}{r} 43 \\ \underline{-\ 7} \end{array}$

3. $\begin{array}{r} 56 \\ \underline{-20} \end{array}$

4. $\begin{array}{r} 544 \\ \underline{-\ 17} \end{array}$

5. $\begin{array}{r} 455 \\ \underline{-250} \end{array}$

6. $\begin{array}{r} 353 \\ \underline{-163} \end{array}$

SOLUTIONS

Octal Complementary Addition

1. $$\begin{array}{r}44\\ -32\\ \hline\end{array} \longrightarrow +45 \longrightarrow \begin{array}{r}44\\ +46\\ \hline 1\,12\end{array}$$

2. $$\begin{array}{r}43\\ -\ 7\\ \hline\end{array} \longrightarrow +70 \longrightarrow \begin{array}{r}43\\ +71\\ \hline 1\,34\end{array}$$

3. $$\begin{array}{r}56\\ -20\\ \hline\end{array} \longrightarrow +57 \longrightarrow \begin{array}{r}56\\ +60\\ \hline 1\,36\end{array}$$

4. $$\begin{array}{r}544\\ -\ 17\\ \hline\end{array} \longrightarrow +760 \longrightarrow \begin{array}{r}544\\ +761\\ \hline 1\,525\end{array}$$

5. $$\begin{array}{r}455\\ -250\\ \hline\end{array} \longrightarrow +527 \longrightarrow \begin{array}{r}455\\ +530\\ \hline 1\,205\end{array}$$

6. $$\begin{array}{r}353\\ -163\\ \hline\end{array} \longrightarrow +614 \longrightarrow \begin{array}{r}353\\ +615\\ \hline 1\,170\end{array}$$

> If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the section "Two's Complement Addition."

Two's Complement Addition

We pointed out previously that the one's complement of a binary number can be found simply by reversing the digits, changing the ones to zeros and zeros to ones:

Number:	1 0 1 1 0 0 0 1
One's Complement:	0 1 0 0 1 1 1 0

A computer can do this *without* subtracting each digit from the value of the base minus one by following two rules:

1. If digit = 0, complement = 1.
2. If digit = 1, complement = 0.

Once the *one's* complement is found, the computer can add 1 to get the radix complement, add this to the minuend, and ignore the overflow just as you did when following the flowchart in Figure 9. By following this process, the computer implements subtraction through addition.

Let's look at an example of complementary addition in binary. Refer to the flowchart in Figure 9 to help you understand the problem below.

$$\begin{array}{r} 101101 \\ -100010 \\ \hline \end{array} \xrightarrow[\text{Complement}]{\text{One's}} +011101 \xrightarrow[\text{One}]{\text{Add}} \begin{array}{r} 101101 \\ +\ 011110 \\ \hline 1\,001011 \end{array}$$

The answer is 1011, which you can check by direct subtraction if you wish.

Here is another example. This one finds:

$011000_2 - 000101_2 =$

$$\begin{array}{r} 011000 \\ -000101 \\ \hline \end{array} \longrightarrow +111010 \longrightarrow \begin{array}{r} 011000 \\ +\ 111011 \\ \hline 1\,010011 \end{array}$$

The answer is 10011_2.

Try the next problem on your own. Check your answer with that on the next page.

$$\begin{array}{r} 110111 \\ -100101 \\ \hline \end{array} \longrightarrow + \qquad \longrightarrow \begin{array}{r} 110111 \\ + \qquad \\ \hline \end{array}$$

$$\begin{array}{r} 110111 \\ -100101 \\ \hline \end{array} \xrightarrow[\text{Complement}]{\text{One's}} +011010 \xrightarrow[\text{One}]{\text{Add}} \begin{array}{r} 110111 \\ +\ 011011 \\ \hline 1\,010010 \end{array}$$

Answer = 10010.

Work one more problem. Then go on to the exercises that follow.

$11101001_2 - 110001_2 =$

Check your work with the solution on the next page.

$$\begin{array}{r} 11101001 \\ -\quad 110001 \\ \hline \end{array} \xrightarrow[\text{Zeros}]{\text{Add}} -00110001 \xrightarrow[\text{Complement}]{\text{One's}}$$

$$\longrightarrow +11001110 \xrightarrow[\text{One}]{\text{Add}} \begin{array}{r} 11101001 \\ +\ 11001111 \\ \hline 1\ 10111000 \end{array}$$

The answer is 10111000_2.

Now go on to the exercises on the next page.

EXERCISES

Two's Complementary Addition

1.
```
   110
  -100
```

2.
```
   110
  -101
```

3.
```
  1111
 -1011
```

4.
```
  1100
 - 101
```

5.
```
 11100
 -  11
```

6.
```
 111001
 - 11110
```

SOLUTIONS

Two's Complementary Addition

1. $\begin{array}{r} 110 \\ -100 \\ \hline \end{array} \longrightarrow +011 \longrightarrow \begin{array}{r} 110 \\ +100 \\ \hline 1010 \end{array}$

2. $\begin{array}{r} 110 \\ -101 \\ \hline \end{array} \longrightarrow +010 \longrightarrow \begin{array}{r} 110 \\ +011 \\ \hline 1001 \end{array}$

3. $\begin{array}{r} 1111 \\ -1011 \\ \hline \end{array} \longrightarrow +0100 \longrightarrow \begin{array}{r} 1111 \\ +0101 \\ \hline 10100 \end{array}$

4. $\begin{array}{r} 1100 \\ -\ 101 \\ \hline \end{array} \longrightarrow +1010 \longrightarrow \begin{array}{r} 1100 \\ +1011 \\ \hline 10111 \end{array}$

5. $\begin{array}{r} 11100 \\ -\ \ \ 11 \\ \hline \end{array} \longrightarrow +11100 \longrightarrow \begin{array}{r} 11100 \\ +11101 \\ \hline 111001 \end{array}$

6. $\begin{array}{r} 111001 \\ -\ 11110 \\ \hline \end{array} \longrightarrow +100001 \longrightarrow \begin{array}{r} 111001 \\ +100010 \\ \hline 1011011 \end{array}$

If you do not understand one or more of these problems, discuss them with your course manager. Otherwise, go on to the next lesson, "Signed Numbers."

Signed Numbers

OBJECTIVES

1. Given a signed decimal or signed octal number, be able to express the number in its equivalent 8- or 16-bit two's complement binary format.

2. Given either an 8- or 16-bit two's complement binary number, be able to convert the number into its octal equivalent.

SAMPLE TEST ITEMS

1. Express in *8-bit two's complement* binary format:

   a. -117_{10} b. $+233_8$

2. Express in *16-bit two's complement* binary format:

   a. 21734_{10} b. 376_8

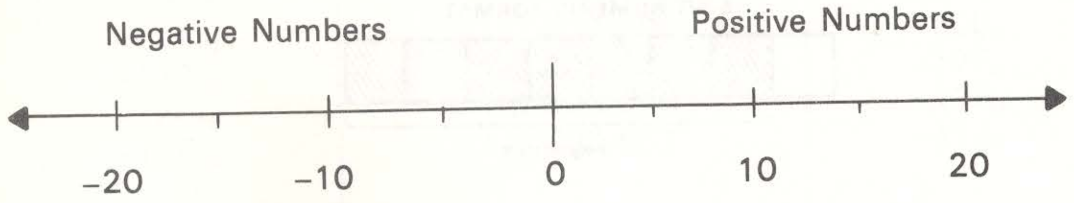
3. Convert from *8-bit two's complement* format to octal:

   a. 00111010 b. 11100111

4. Convert from *16-bit two's complement* format to octal:

   a. 1110111101000111 b. 0011000111010100

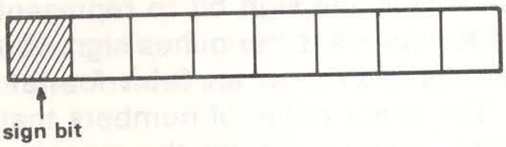
Signed numbers allow us to work with both positive and negative values. Positive numbers are *greater* than 0 and are indicated by the *plus* (+) sign. Negative numbers are *less* than 0 and are indicated by the *minus* (–) sign.



When working with numbers, computers do not store the plus and minus signs in their original forms, + and –. Instead, they incorporate the sign in the number by arranging the bits in special ways. Each method for storing signed numbers is called a *numeric format.* It is important to understand numeric formats so that you can interpret the contents of a computer's memory locations. People working with computer hardware examine memory locations to make sure that the computer's circuits are working properly. People working with computer software examine memory locations to determine the effects of their programs. This lesson explains the numeric format most often used by minicomputers by showing its advantages over two simpler formats.

All numeric formats use one bit to indicate the sign of a number. This special bit is called a *sign bit* and is usually the *leftmost* bit in the format.

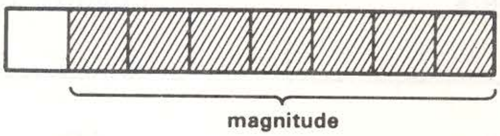
8-BIT NUMERIC FORMAT



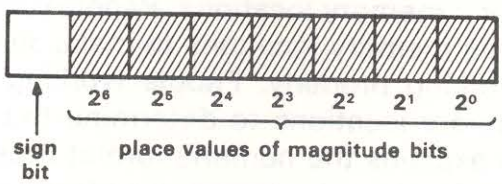
NOTE

We are specifically using the word "format" rather than "word" because some computers use numeric formats that span more than one computer word. The important factor is only the number of bits in the format, not the number of words needed to make up that format.

Since one bit is required to indicate the sign, seven bits are left in an 8-bit numeric format. These bits are used to express the *magnitude* of the number.



Given the number of bits in the magnitude of a numeric format, we can compute the maximum value of the magnitude. In an 8-bit format, for example, the place values of the bits are as follows:



The maximum value of the magnitude is thus 1111111_2, or $2^7 - 1$, which is 127_{10}. If we let "N" equal the number of bits in a numeric format, we can represent the maximum value of the magnitude as $2^{(N-1)} - 1$.

Most computers use a 0 in the sign bit to represent the plus sign, and a 1 in the sign bit to represent the minus sign. With this convention, the value of a number stored in an 8-bit format can vary from about +127 to −127. The exact range of numbers that can be stored depends on the technique used to express the magnitude of the number.

The most common method of expressing positive and negative binary numbers is with the two's complement format shown in Table 3. Notice four important points:

1. Each negative number is expressed as the two's complement of the corresponding positive number

2. Zero is always a positive number.

Table 3
Numbers in 8-bit Two's Complement Format

Number in Decimal	Number in Binary	Number in Octal
-128_{10}	1 0 000 000	-200_8
-127_{10}	1 0 000 001	-177_8
-126_{10}	1 0 000 010	-176_8
-125_{10}	1 0 000 011	-175_8
•	•	•
•	•	•
•	•	•
-2_{10}	1 1 111 110	-2_8
-1_{10}	1 1 111 111	-1_8
0_{10}	0 0 000 000	0_8
$+1_{10}$	0 0 000 001	$+1_8$
$+2_{10}$	0 0 000 010	$+2_8$
•	•	•
•	•	•
•	•	•
$+125_{10}$	0 1 111 101	$+175_8$
$+126_{10}$	0 1 111 110	$+176_8$
$+127_{10}$	0 1 111 111	$+177_8$

3. There is one more negative number than there are positive numbers.

4. When 1 is added to the highest number in this format, $+127_{10}$, the answer is the lowest number, -128_{10}:

$$\begin{array}{rcl} +127_{10} & = & +0\ 1\ 111\ 111 \\ +\quad 1_{10} & = & +0\ 0\ 000\ 001 \\ \hline & & 1\ 0\ 000\ 000 \quad = -128_{10} \end{array}$$

Here is an example of two's complement addition

$$\begin{array}{rcc} -12_8 & = & 11110100 \\ +24_8 & = & +00011000 \\ \hline +12_8 & = & 100001100 \\ & & \text{(ignore overflow)} \end{array}$$

With 8 bits it is possible to express all numbers in the range -128_{10} to $+127_{10}$. To increase the range of numbers, a longer format is necessary, for example, 16 or even 32 bits. A range of -2^{15} to $+2^{15}-1$, or -32768_{10} to $+32767_{10}$ is allowed by 16-bit two's complement format. A range of -2^{31} to $+2^{31}-1$, or -2147483648_{10} to $+2147483647_{10}$ is allowed by 32-bit two's complement format. If these ranges are still not large enough, or if we want to work with fractional numbers, we usually use a different type of format.

There are other formats used when expressing signed numbers in binary. Each of these formats has distinct advantages and disadvantages. Two's complement format is the format most commonly used to express signed numbers in binary because it has the most advantages.

EXERCISES

Two's Complement Format

1. Express each of the following numbers in *8-bit* two's complement format. (Watch the bases!)

   a. $+34_{10}$

   b. -91_{10}

   c. -74_8

   d. $+36_8$

2. Express each number below in *16-bit two's* complement format.

   a. $+10021_{10}$

   b. -6398_{10}

   c. $+5051_8$

   d. -22016_8

SOLUTIONS

Two's Complement Format

1. 8-bit format:

a.	$+34_{10}$	=	$+42_8$	=	00100010
b.	-91_{10}	=	-133_8	=	10100101
c.			-74_8	=	11000100
d.			$+36_8$	=	00011110

2. 16-bit formats:

a.	$+10021_{10}$	=	$+23445_8$	=	0 010 011 100 100 101
b.	-6398_{10}	=	-14376_8	=	1 110 011 100 000 010
c.			$+5051_8$	=	0 000 101 000 101 001
d.			-22016_8	=	1 101 101 111 110 010

3. Convert the numbers below from 8-bit *two's* complement format to octal.

   a. 00110111

   b. 11111111

   c. 11110000

   d. 00010101

4. Convert the following numbers from 16-bit *two's* complement format to octal.

   a. 1011111100001101

   b. 0000111110101100

   c. 0110010111000101

   d. 1101110001010101

3. 8-bit to octal:

   a. 0 0 110 111 = 67_8

   b. 1 1 111 111 → Two's Complement → 0 0 000 001 → $-\ 1_8$

   c. 1 1 110 000 → Two's Complement → 0 0 010 000 → -20_8

   d. 0 0 010 101 = 25_8

4. 16-bit to octal:

   a. 1 011 111 100 001 101 → 0 100 000 011 110 011 → -40363_8

   b. 0 000 111 110 101 100 = $+07654_8$

   c. 0 110 010 111 000 101 = $+62705_8$

   d. 1 101 110 001 010 101 → 0 010 001 110 101 011 → -21653_8

> If you do not understand the solutions to one or more of the problems in the previous four exercises, discuss them with your course manager. Otherwise, go on to the Module Review.

Module Review

This module has shown you how computers do arithmetic in binary by using only the addition operation. It taught you to add and subtract in octal and binary while still "thinking" in decimal, and you learned how to do complementary addition in all three bases. These skills will help you understand how and why computers work the way they do.

The test for this module asks you to demonstrate the skills that you have learned by doing the tasks listed below. (Sample test items for these tasks are listed at the beginning of each lesson.)

1. Add numbers in octal and binary.
2. Subtract directly in octal and binary.
3. Subtract using complementary addition in octal and binary.
4. Express signed decimal and octal numbers in *two's* complement binary formats.
5. Convert numbers from *two's* complement binary formats into octal.

If you are unsure of your ability in one or more of these areas, go back and review the related exercises, doing any extra problems that you have not yet completed. If you have already completed all the related exercises, ask your course manager for additional help.

When you are ready, ask your course manager for a copy of the test on Computer Arithmetic. Work the problems and then check your work against the corresponding Evaluation Sheet.