Reference 1 Abridged English Translation

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Longitudinal Vibration of Short Circular Cylinders

by

Issac Koga (Tokyo University of Engineering)

Summary

In order to investigate precisely the vibration of piezoelectric elements, it is very important to treat theoretically the longitudinal vibration of short cylinders. At present time, strict solution for this problem does not appear in any journal, and only approximate treatments are known in which a smaller sectional area compared to its length is assumed. The results of these studies are often not satisfactory for the calculation of vibration frequencies. Vibration modes obtained from the analysis are different from those estimated previously.

Therefore, I intend to introduce strict theories to treat the vibration of short circular cylinders made of isotropic crystals. The results of this analysis give several significant new facts on the vibration. Although it is highly difficult to apply this method to anisotropic crystal such as quartz, the result of this study will be helpful to explain the vibration of anisotropic crystal qualitatively.

1. Introduction

Recently applications of quartz crystal oscillation are widely spread in the fields of high frequency technologies. Many experiments have been done throughout the world and empirical rules for calculation of oscillation frequencies from dimensions of plates are known. These rules are sometimes practically out of agreement with the experimental data. In some cases, oscillation itself is impossible. Various studies were made to clarify the behavior of vibration. As an example, Lack introduced ideas using equivalent representation of oscillation modes in crystal. However, most of these studies did not make clear the behavior of plates.

I will intend to establish strict theoretical treatment of crystal. However, theory of quartz as anisotropic crystal is still very complicated problem, then, as the first step, this study puts assumption that the crystal is isotropic, and for simplicity, plates are restricted in circular cylinders.

2. Lateral Vibration of Long and Short Cylinders

Previously the discussions on lateral vibration of circular cylinders were made, according to the practical requirements, for cylinders having small sectional areas compared to its length. Inversely, for thin quartz crystal plate, length is smaller compared to its sectional size. As far as I know, study for this type of crystal plates is not reported.

Frequency (fundamental period) of vibration of long cylinders with length 2ℓ is already known for thickness-mode vibration, as following formula (1).

$$\frac{1}{4l}\sqrt{\frac{E}{\rho}}\times\frac{2\pi}{p}=1 \qquad \text{------------------------------ (1)}$$

Here, (2 π / p) is fundamental period (p is angular frequency 2 π f), E is elastic constant, ρ is density of material. In some cases, this formula gives well approximation for this plate. However, vibration of a thin plate (for example, Curie-cut plate) must be treated by the combination of two vibration modes (thickness and radial modes).

3. Equation of Motion of Lateral Vibration of Short Circular Cylinders

Basic equations of motion of elastic bodies, using cylindrical coordinate (r, Θ, z) are shown as eq. (2).

$$\left.\begin{aligned}
\rho\frac{\partial^2 u}{\partial t^2}&=(\lambda+2\mu)\frac{\partial\phi}{\partial r}-\frac{2\mu}{r}\frac{\partial\psi_z}{\partial\theta}+2\mu\frac{\partial\psi_\theta}{\partial z}\\
\rho\frac{\partial^2 v}{\partial t^2}&=(\lambda+2\mu)\frac{1}{r}\frac{\partial\phi}{\partial\theta}-2\mu\frac{\partial\psi_r}{\partial z}+2\mu\frac{\partial\psi_z}{\partial r}\\
\rho\frac{\partial^2 w}{\partial t^2}&=(\lambda+2\mu)\frac{\partial\phi}{\partial z}-\frac{2\mu}{r}\frac{\partial}{\partial r}(r\psi_\theta)+\frac{2\mu}{r}\frac{\partial\psi_r}{\partial\theta}
\end{aligned}\right\} \qquad \text{------------- (2)}$$

In this equation, (u, v, w) in variables ψ and φ are displacements for (r, Θ, z) directions respectively. Using Poisson-ratio σ, constants λ and μ are decided as equation (3).

By applying boundary conditions of both ends, eq. (2) are simplified as (4) to (7)

In order to solve these differential equations, assume that the waves are sinusoidal form of

$$\phi\ ,\ \phi\ :\ \exp(ipt) \qquad \text{-- (10)}$$

As the result, solutions which must be solved are (11) and (12) with (13),

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{\partial^2\phi}{\partial z^2}+h^2\phi=0 \qquad \text{------------------------------- (11)}$$

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}-\frac{\psi}{r^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0 \qquad \text{--------------------- (12)}$$

$$h^2=\frac{p^2\rho}{\lambda+2\mu},\qquad k^2=\frac{p^2\rho}{\mu} \qquad \text{--- (13)}$$

Next, by introducing Separation of Variables as

$$\phi\ ,\ \phi\ =\ R(r)\cdot Z(z) \qquad \text{--- (14)}$$

Then major equations (11) and (12) are transformed to analytically solvable forms like (15).

Consequently, solutions of wave motion ψ and φ are expressed using Bessel function:

$$\phi = AJ_0(\xi r)\cosh \alpha z \cdot e^{ipt} \quad \text{-- (16)}$$

$$2\psi = B\frac{\partial J_0(\xi r)}{\partial r}\sinh \beta z \cdot e^{ipt} \quad \text{------------------------------------ (17)}$$

$$\alpha^2 = \xi^2 - h^2, \quad \beta^2 = \xi^2 - k^2 \quad \text{-- (18)}$$

By applying boundary conditions with technical transformation of equations, needed general expressions are obtained as eq. (30) and (32)

$$\frac{\tanh \beta l}{\tanh \alpha l} = \frac{4\alpha\beta\xi^2}{(\xi^2+\beta^2)^2} \quad \text{-- (30)}$$

$$\frac{d^2 J_0(\xi a)}{d(\xi a)^2} + \frac{\xi^2+\beta^2}{2\xi^2} J_0(\xi a) = 0 \quad \text{----------------------------------- (32)}$$

4. Fundamental Period of Vibration

To calculate fundamental period (actual frequency) of vibration, it is needed to eliminate variables α, β and ξ from equations (18) (30) and (32). It is not easy to solve these simultaneous equations, except for introducing special relations. By referring Lamb's technique (ref. 4), proposed steps are: through equations (18) and (30), get relations of (ξ / k) and (ξ ℓ), and combine with (ξ a) obtained from equation (32).

Consequently, when the radius α of cylinder is kept constant, relation of (angular) frequency p and the length (thickness) ℓ is obtained as equation (33) and when the length is kept constant, relation between frequency and the radius a is given as equation (34).

$$\left.\begin{aligned} \frac{1}{kl} &= \frac{2}{2l}\sqrt{\frac{\mu}{\rho}} \times \frac{1}{p} \\ \frac{1}{\xi l} &= \frac{1}{(\xi a) l} \times a \end{aligned}\right\} \quad \text{------------------------------- (34)}$$

Concrete so ... on Ratio σ (ex. 1/3) are (see Appendix I, II)

$$(\xi \cdot a) = 1.81, \quad 5.34, \quad 8.54, \quad 11.70, \quad \cdots \quad \text{------------- (37)}$$

In order to get solution of wanted variable (which finally determine frequency) from given dimensions of plate, equations (38) and (39) are usable with conventional diagrams are shown in Fig1, 2 and 3.

$$\frac{1}{2\pi a}\sqrt{\frac{1}{\rho} \cdot \frac{E}{1-\sigma^2}} \times \frac{2\pi}{p} = \frac{1}{(\xi a)}. \quad \text{------------------ (38)}$$

$$\frac{1}{4l}\sqrt{\frac{2\mu}{\rho}} \times \frac{2\pi}{p} = 1 \quad \text{-- (39)}$$

5. Relation between Frequency and Temperature

In practical applications of vibration plates, effects of temperature to the frequencies of oscillation are often important. For discussing these effects, it is possible to use equation (40) with results obtained above.

6. Vibration Modes

Actual modes of vibration can also be discussed by using discussion above (with Appendix III)
To treat complicated general case, we will show some typical cases here.
Examples of distribution of amplitude of obtained modes are shown in Fig 4. ("q" is arbitrary integer)

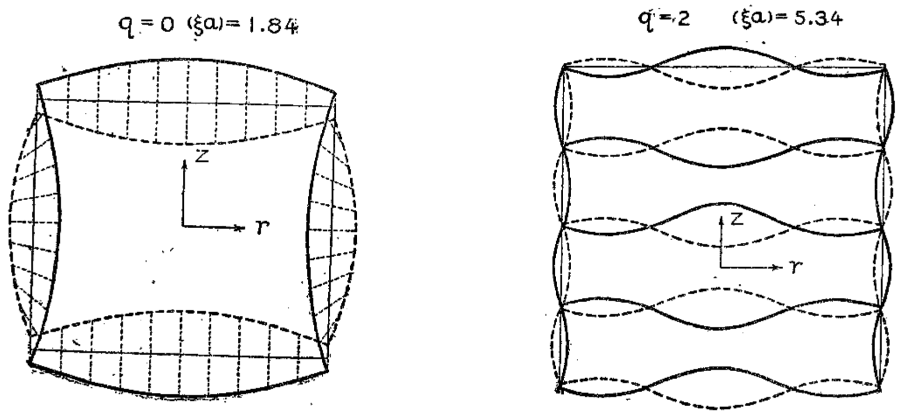

Figure 4 Examples of distribution of amplitude of obtained modes

7. Conclusions

1) Vibration of short circular cylinders, shown in Fig 1 to 3 as examples, can be thought to be comprising from two modes. First one is largely dependent to length of cylinders but less dependent on diameter. Inversely, second mode is largely dependent to diameter and less dependent to cylinder length. They may be called as Axial and Radial Mode respectively.

2) When length of cylinders is very small compared with its diameter (thin plates), frequencies of the Radial mode can be calculated by eq. (38), and, frequency of the Axial mode can be gotten from eq. (39) instead of previously used formula (1).

3) Example of Axial mode is shown in Fig 4.

4) Previously, in a Curie-cut oscillation plate, three dominant frequencies are observed, but details of the intermediate frequency are not well clarified. By discussions explained above, it may be concluded without doubt that this vibration can be interpreted as the Radial mode. Previously, from practical reason, investigation for long cylinders were well discussed. However, for the application to a thin quartz oscillation plate, studies on short cylinders become very important.

8. References

1. F.R. Lack: "Observations on Modes of Vibration and Temperature Coefficients of Quartz Crystal Plates", IRE 17 p. 1123 (July 1929), BSTJ 8 p.515 (July 1929), IEE Japan, News p.32 (Jan 1930)

2. Load Rayleigh: "The Theory of Sound", Vol. l Chap.VII, Longitudinal and Torsional Vibration of Bars, p.242-254
3. A. E. H. Love : "A Treatise on the Mathematical Theory of Elasticity", 4th Ed. Chap. XII, Vibration of Spheres and Cylinders, P.289-290, Chap. XX, Vibration of Rods, Problems of Dynamical Resistance p.428
4. H. Lamb:" On Waves in an Elastic Plate", Proc. Royal Society London, 93, No. A, 648 p. 114 ~ 128 (Mar. 1, 1917)

9. Appendices

I. Relation of (ξ / k) between (ξ ℓ), and (1/kℓ) between (1/$\xi\ell$)

Expanded discussion from Lamb's results in Ref. 4. Detailed tables on calculated values of solutions (ξ / k) and (ξ ℓ) are equipped (Table 1 ~ 3) with two Figures.

Π. Discussion on two solution equations (32) and (36)

Detailed data compared are calculated as in Table 4and Fig 6.

III Displacement of crystal waves in the special case (ξ / k) =1/2

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